

A SPARSE GRID COLLOCATION METHOD FOR PARABOLIC PDES WITH RANDOM DOMAIN DEFORMATIONS

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ABSTRACT. This work considers the problem of numerically approximating statistical moments of a Quantity of Interest (QoI) that depends on the solution of a time dependent linear parabolic partial differential equation. The geometry is assumed to be random and is parameterized by N random variables. The parabolic problem is remapped to a fixed deterministic domain with random coefficients and shown to admit an extension to a well defined region in \mathbb{C}^N with respect to the random variables. To compute the stochastic moments of the QoI, a Smolyak sparse grid stochastic collocation method is used. To confirm the convergence rates, a comparison to numerical experiments is performed.

1. INTRODUCTION

Mathematical modeling forms an essential part for understanding many engineering and scientific applications with physical domains. These models have been widely used to predict QoI of any particular problem when the underlying physical phenomena is well understood. However, in many cases the practicing engineer or scientist does not have direct access to the underlying geometry and uncertainty is introduced. It is now essential to quantify the influence of the domain uncertainty on the Quantities of Interest.

In this work a method to efficiently solve parabolic PDEs with moderate random geometrical deformation is introduced. Application examples include subsurface aquifers with soil variability diffusion problems, ocean wave propagation (sonar) with geometric uncertainty, chemical diffusion with uncertain geometries, among others.

Collocation and perturbation approaches have been developed to quantify the statistics of the QoI for elliptic PDEs with random domains. The perturbation approaches [10, 20] are accurate only for small domain perturbations. In contrast, the collocation approaches [3, 7, 19] allow the computation of the statistics for larger domain deviations, but lack a full error convergence analysis. Recently, in [4], the authors present a collocation approach for elliptic PDEs based on Smolyak grids. An analyticity analysis is done and convergence rates are derived. This work is extended, in part, in [9].

This paper is a extension of analysis and error estimates derived in [4] to parabolic PDEs. The reader is encouraged to tackle that paper first. Moreover, for simplicity much of the notation is kept the same.

In this work a rigorous convergence analysis of the collocation approach based on isotropic Smolyak grids for parabolic PDEs is also presented. This consists of an analysis of the regularity of the solution with respect to the stochastic domain parameters. It is then shown that the solution can be analytically extended to a well defined region $\Theta_\beta \subset \mathbb{C}^N$ with respect to the domain random variables. To simplify the proof it is shown first that the solution is analytic along each separate dimension and then Hartog's Theorem [14] is applied to show that the solution is analytic in all of

Θ_β . Furthermore, error estimates are derived both in the “energy norm” as well as on functionals of the solution for Clenshaw Curtis abscissas that can be easily generalized to a larger class of sparse grids.

The outline of the paper is the following: In Section 2 the mathematical problem setup is discussed. The random domain parabolic PDE problem is remapped onto a deterministic domain with random matrix coefficients. The random boundary is parameterized by N random variables. In Section 3 the solution is shown to be analytically extended to $\Theta_\beta \subset \mathbb{C}^N$. In Section 4 Smolyak sparse grids are introduced. In Section 5 error estimates for the mean and variance of the QoI with respect to the sparse grid and truncation approximations are derived. Finally, in section 7 numerical examples are presented.

2. SETUP AND PROBLEM FORMULATION

Let Ω be the set of outcomes from the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathcal{F} is a sigma algebra of events and \mathbb{P} is a probability measure. Define $L_P^q(\Omega)$, $q \in [1, \infty]$, as the space of random variables such that

$$L_P^q(\Omega) := \{v \mid \int_{\Omega} |v(\omega)|^q d\mathbb{P}(\omega) < \infty\} \text{ and } L_P^\infty(\Omega) := \{v \mid \text{ess sup}_{\omega \in \Omega} |v(\omega)| < \infty\},$$

where $v : \Omega \rightarrow \mathbb{R}$ be a measurable random variable.

Suppose $D(\omega) \subset \mathbb{R}^d$ is an open bounded domain with Lipschitz boundary $\partial D(\omega)$ parameterized with respect to a stochastic parameter $\omega \in \Omega$. Now, consider the following stochastic Parabolic boundary value problem: Given that $f(\cdot, t, \omega) : D(\omega) \times [0, T] \rightarrow \mathbb{R}$, $a(\cdot, \omega) : D(\omega) \rightarrow \mathbb{R}$ and $u_0 : D(\omega) \rightarrow \mathbb{R}$ are continuous with bounded covariance functions find $u : D(\omega) \times [0, T] \rightarrow \mathbb{R}$ such that almost surely

$$\begin{aligned} \partial_t u(x, t, \omega) - \nabla \cdot (a(x, \omega) \nabla u(x, t, \omega)) &= f(x, t, \omega), \text{ in } D(\omega) \times [0, T] \\ u(0, t, \omega) &= 0 \quad \text{on } \partial D(\omega) \times [0, T] \\ u(x, t, \omega) &= u_0 \quad \text{on } D(\omega). \end{aligned}$$

The diffusion coefficient satisfies the following assumption.

Assumption 1. *There exist constants a_{min} and a_{max} such that*

$$0 < a_{min} \leq a(x, \omega) \leq a_{max} < \infty \text{ for a.e. } x \in D(\omega), \omega \in \Omega,$$

where $a_{min} := \text{ess inf}_{x \in D(\omega), \omega \in \Omega} a(x, \omega)$ and $a_{max} := \text{ess sup}_{x \in D(\omega), \omega \in \Omega} a(x, \omega)$.

The weak formulation is stated as:

Problem 1. *Find $u(\cdot, t, \omega) \in L^2(0, T; H_0^1(D(\omega)))$ s.t.*

$$\begin{aligned} (1) \quad \int_{D(\omega)} \partial_t uv + a \nabla u \cdot \nabla v \, dx &= \int_{D(\omega)} f v \, dx \quad \forall v \in H_0^1(D(\omega)) \text{ in } D(\omega) \times [0, T] \\ u(0, t, \omega) &= 0 \quad \text{on } \partial D(\omega) \times [0, T] \\ u(x, t, \omega) &= u_0 \quad \text{on } D(\omega) \end{aligned}$$

almost surely in Ω , where $f(\cdot, t, \omega) \in L^2(0, T; L^2(D(\omega)))$ and $u_0 \in L^2(D(\omega))$.

Under Assumption 1 the weak formulation has a unique solution up to a zero-measure set in Ω (see [6]).

2.1. Reformulation onto a fixed Domain. In this section the stochastic Problem 1 is reformulated to a fixed domain with random coefficients.

Now, for every $\omega \in \Omega$ set $D(\omega)$ as the image of the map $F : U \times \Omega \rightarrow D(\omega)$, where the domain $U \subset \mathbb{R}^d$ is a fixed Lipschitz domain and does not depend on the stochastic parameter ω . The following assumption is made, which is slightly modified from Assumption 2 in [4].

Assumption 2. *Given the one-to-one map $F : U \times \Omega \rightarrow \mathbb{R}^d$ there exist constants \mathbb{F}_{\min} and \mathbb{F}_{\max} such that $0 < \mathbb{F}_{\min} \leq \sigma_{\min}(\partial F(\omega))$ and $\sigma_{\max}(\partial F(\omega)) \leq \mathbb{F}_{\max} < \infty$. almost everywhere in U and almost surely in Ω . Denoted by $\sigma_{\min}(\partial F(\omega))$ (and $\sigma_{\max}(\partial F(\omega))$) the minimum (respectively maximum) singular value of the Jacobian $\partial F(\omega)$. In addition, for the rest of the paper the terms a.s. and a.e. will be dropped unless emphasis or disambiguation is needed.*

By applying a change of variables and the chain rule Problem 1 can be reformulated to the fixed domain U . First, observe that for any $v \in C^1(D(\omega))$

$$(2) \quad \nabla v = \partial F^{-T} \nabla(v \circ F).$$

It follows that Problem 1 is reformulated as

Problem 2. *Given that $f(\cdot, t, \omega) \in L^2(0, T; L^2(U))$ and $u_0 \circ F \in L^2(U)$ find $u \circ F \in L^2(0, T; L^2(U))$ s.t.*

$$(3) \quad \begin{aligned} \int_U \partial_t u \circ F v + B(\omega; u \circ F, v) &= l(\omega; v), \quad \text{in } U \times [0, T] \\ u \circ F &= 0 \quad \text{on } \partial U \times [0, T] \\ u \circ F &= u_0 \circ F \quad \text{on } U \times \{t = 0\} \end{aligned}$$

almost surely $\forall v \in H_0^1(D(\omega))$, where for any $v, s \in H_0^1(D(\omega))$ and $C = \partial F^T \partial F$ we have that

$$\begin{aligned} B(\omega; s, v) &:= \int_U (a \circ F(\cdot, \omega) \nabla s^T C^{-1}(\cdot, \omega) \nabla v |\partial F(\cdot, \omega)|), \\ l(\omega; v) &:= \int_U (f \circ F(\cdot, \omega)) v |\partial F(\cdot, \omega)|, \end{aligned}$$

Lemma 1. *Under Assumptions 2 it is immediate to see that*

- i) $L^2(D(\omega))$ and $L^2(U)$ are isomorphic.
- ii) $H^1(D(\omega))$ and $H^1(U)$ are isomorphic.
- iii) $L^2(0, T; L^2(D(\omega)))$ and $L^2(0, T; L^2(U))$ are isomorphic.
- iv) $L^2(0, T; H^1(D(\omega)))$ and $L^2(0, T; H^1(U))$ are isomorphic.

2.2. Mixed Boundary Conditions. For many practical applications the mixed Dirichlet and Neumann boundary value problem is more relevant. Suppose the boundary $\partial U = \partial U_1 \cup \partial U_2$, where ∂U_1 and ∂U_2 are disjoint. Similarly, let $\partial D(\omega) = \partial D_1(\omega) \cup \partial D_2(\omega)$

Now, consider the follow boundary value problem: Given that $f(\cdot, t, \omega) : D(\omega) \times [0, T] \rightarrow \mathbb{R}$, $a(\cdot, \omega) : D(\omega) \rightarrow \mathbb{R}$ are continuous with bounded covariance functions, find $u : D(\omega) \times [0, T] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \partial_t u(x, t, \omega) - \nabla \cdot (a(x, \omega) \nabla u(x, t, \omega)) &= f(x, t, \omega) & \text{in } D(\omega) \times [0, T] \\ u(x, t, \omega) &= g_1(x) & \text{on } \partial D_1(\omega) \times [0, T] \\ a(x, \omega) \nabla u(x, t, \omega) \cdot \nu &= g_2(x) & \text{on } \partial D_2(\omega) \times [0, T] \\ u(x, 0, \omega) &= u_0(x) & \text{on } D(\omega) \times \{t = 0\} \end{aligned}$$

almost surely, where $g_1, g_2 \in C(\partial D(\omega))$ are bounded and ν is the outward unit norm vector along $\partial D(\omega)$. From [13] the problem is posed as: Given that $f(\cdot, t, \omega) \in L^2(0, T; L^2(D(\omega)))$ and $g_1, g_2 \in H^{1/2}(\partial D(\omega))$ find $u \in L^2(0, T; H^1(D(\omega)))$ s.t.

$$\begin{aligned} \int_{D(\omega)} \partial_t u \circ F v + a(x, \omega) \nabla u \cdot \nabla v + \int_{D_2(\omega)} u v &= l(\omega; v), & \text{in } D(\omega) \times [0, T] \\ u(x, t, \omega) &= 0 & \text{on } \partial D_1(\omega) \times [0, T] \\ u(x, 0, \omega) &= u_0(x) & \text{on } D(\omega) \times \{t = 0\} \end{aligned}$$

$\forall v \in H_0^1(D(\omega))$ almost surely, where

$$l(\omega; v) := \int_U f(x, t, \omega) v - \int_{\partial D_2(\omega)} g_2 v - \int_{D(\omega)} a(x, \omega) \nabla \mathbf{w} \cdot \nabla v$$

and $T_D : H^{1/2}(\partial D(\omega)) \rightarrow H^1(D(\omega))$ is a linear bounded operator such that $\forall \hat{g} \in H^{1/2}(\partial D(\omega))$, $\mathbf{w} := T_D \hat{g} \in H^1(D(\omega))$ satisfies $\mathbf{w}|_{\partial U} = \hat{g}$ almost surely.

Now, since $\partial D(\omega)$ is a Lipschitz boundary then it can be described by the implicit function $S_{\partial D}(\omega) = 0$ for all $x \in \partial D(\omega)$. Now, assume that $F : \partial U \rightarrow \partial D(\omega)$, thus $S_{\partial D(\omega)} \circ F = S_{\partial U} = 0$. It follows that $\nabla S_{\partial D(\omega)} = \partial F^{-T} \nabla S_{\partial D(\omega)} \circ F = \partial F^{-T} \nabla S_{\partial U}$, thus

$$\nu = \frac{\nabla S_{\partial D}(\omega)}{|\nabla S_{\partial D}(\omega)|} = \frac{\partial F^{-T} \nabla S_{\partial D(\omega)} \circ F}{|\partial F^{-T} \nabla S_{\partial D(\omega)} \circ F|} = \frac{\partial F^{-T} \nabla S_{\partial U}}{|\partial F^{-T} \nabla S_{\partial U}|} = \frac{|\nabla S_{\partial U}|}{|\partial F^{-T} \nabla S_{\partial U}|} \partial F^{-T} \nu_U,$$

where ν_U is the unit outward normal from the boundary ∂U . The Neumann boundary condition $a(x, \omega) \nabla u(x, t, \omega) \cdot \nu = g_2(x)$ for all $x \in \partial D(\omega)_2$ is re-mapped as

$$(a \circ F) \partial F^{-T} \nabla \tilde{u} \circ F \cdot \partial F^{-T} \nu_U = \frac{|\partial F^{-T} \nabla S_{\partial U}| g_2 \circ F}{|\nabla S_{\partial U}|}$$

for all $x \in \partial U_1$. Now, let $V := \{v \in H^1(U) : v = 0 \text{ on } \partial U_1\}$, then remap the mixed boundary value problem as:

Problem 3. *Given that $f(\cdot, t, \omega) \in L^2(0, T; L^2(U))$ and $g_1 \circ F, g_2 \circ F \in H^{1/2}(\partial U)$ find $\tilde{u} \circ F \in L^2(0, T; V)$ s.t.*

$$\begin{aligned} \int_U \partial_t \tilde{u} \circ F v + B(\omega; \tilde{u} \circ F, v) &= l(\omega; v), & \text{in } U \times [0, T] \\ \tilde{u} \circ F &= 0, & \text{on } \partial U_1 \times [0, T] \\ \tilde{u}(x, 0, \omega) &= u_0(x) & \text{on } U \times \{t = 0\} \end{aligned}$$

$\forall v \in V$ almost surely, where

$$\tilde{l}(\omega; v) := \int_U f \circ F v - \int_{\partial U_2} \frac{\det(\partial F) |\partial F^{-T} \nabla S_{\partial U}| g_2 \circ F}{|\nabla S_{\partial U}|} v - B(\omega; \mathbf{w} \circ F, v)$$

The weak solution $u \circ F \in H^1(U)$ for the non-zero Dirichlet boundary value problem is simply obtained as $u \circ F = \tilde{u} \circ F + \mathbf{w}$.

2.3. Quantity of Interest. In practice one is interested in computing the statistics of a Quantity of Interest (QoI) over the stochastic domain or a subdomain of it. Consider QoI of the form

$$(4) \quad Q(u) := \int_D q(u(t) \circ F) |\partial F| \, dx$$

($q \in C^\infty(U) \cap L^2(U)$) for $T \geq t > 0$ over the region $\bar{D} \subset U$. Furthermore assume that a mapping F s.t. $\partial F|_{\bar{D}} = I$ can be constructed so that $\bar{U} = F^{-1}(\bar{D})$ does not depend on the parameter $\omega \in \Omega$.

In this paper our attention is restricted to the computation of the mean $\mathbb{E}[Q]$ and variance $\text{Var}[Q] := \mathbb{E}[Q^2] - \mathbb{E}[Q]^2$ given that the domain deformation is parameterized by a stochastic random vector.

Assume that $Q : H_0^1(U) \rightarrow \mathbb{R}$ is a bounded linear functional. The influence function is computed as from the dual equation as

Problem 4. Find $\varphi \in L^2(0, T; V)$ such that $\forall v \in V$

$$(5) \quad \int_U \partial_t \varphi v + B(\omega; v, \varphi) = Q(v)$$

a.s. in Ω where $\varphi(x, 0, \omega) = 0$.

Thus the QoI can be computed as $Q(u) = \int_U u \partial_t \varphi + B(\omega; u, \varphi)$. Now, assume that $\text{dist}(\bar{D}, \partial D) \geq \delta$ for some $\delta > 0$. Now, pick $\mathbf{w} = T_D g_2$ such that $Q(\mathbf{w}) = 0$, thus

$$Q(u \circ F) = Q(\tilde{u} \circ F) = \int_U \tilde{u} \circ F \partial_t \varphi + B(\omega; \tilde{u} \circ F, \varphi).$$

2.4. Domain Parameterization. The mapping $F(\cdot, \omega) : U \rightarrow D(\omega)$ is assumed to be represented as a finite expansion of the finite random variables Y_1, \dots, Y_N .

Let $Y := [Y_1, \dots, Y_N]$ be a N valued random vector measurable in $(\Omega, \mathcal{F}, \mathbb{P})$ taking values on $\Gamma := \Gamma_1 \times \dots \times \Gamma_N \subset \mathbb{R}^N$ and $\mathcal{B}(\Gamma)$ be the Borel σ -algebra. Define the induced measure μ_Y on $(\Gamma, \mathcal{B}(\Gamma))$ as $\mu_Y := \mathbb{P}(Y^{-1}(A))$ for all $A \in \mathcal{B}(\Gamma)$. Assuming that the induced measure is absolutely continuous with respect to the Lebesgue measure defined on Γ , then there exists a density function $\rho(\mathbf{y}) : \Gamma \rightarrow [0, +\infty)$ such that for any event $A \in \mathcal{B}(\Gamma)$ $\mathbb{P}(Y \in A) := \mathbb{P}(Y^{-1}(A)) = \int_A \rho(\mathbf{y}) \, d\mathbf{y}$. For any measurable function $Y \in L_P^1(\Gamma)$ the expected value is defined as $\mathbb{E}[Y] := \int_\Gamma \mathbf{y} \rho(\mathbf{y}) \, d\mathbf{y}$ and define $L_P^q(\Omega)$ as the functional space $L_P^q(\Omega)$ with respect to the density function $\rho(\mathbf{y})$.

The mapping $F(\cdot, \omega) : U \rightarrow D(\omega)$ can be parameterized in many forms. In this paper our attention is restricted to the following class of mappings:

Suppose that $U_i \subset U \subset \mathbb{R}^d$, $i = 1, \dots, M$, is a collection of non overlapping open elements, such as square, triangular, tetrahedral, nurbs, etc, in \mathbb{R}^d . In addition, let $U := \cup_{i=1}^M \bar{U}_i$ form a Lipschitz bounded domain. For each element U_i , $i = 1, \dots, M$ suppose $F_i : U_i \times \Omega \rightarrow D_i(\omega)$ satisfies Assumption 2. Denote $D(\omega) := \cup_{i=1}^M D_i(\omega)$ (see Figure 1) and assume that it forms a conformal mesh.

Assumption 3. For each open element $U_i \subset \mathbb{R}^d$, $i = 1, \dots, M$, the map $F_i : U_i \times \Omega \rightarrow \mathbb{R}$ has the form $F_i(x, \omega) := x + q_i(x, \omega)$, where $q_i(x, \omega) := e_i(x, \omega) \hat{v}_i(x)$ a.s. in Ω , with $\hat{v}_i : U_i \rightarrow \mathbb{R}^d$, $\hat{v}_i \in C^\infty(U_i)$, and $e_i(x, \omega) : U_i \times \Omega \rightarrow \mathbb{R}$.

The next step is to characterize the stochastic perturbation variables e_1, \dots, e_M . Without loss of generality characterize only a single stochastic perturbation $e(x, \omega) : \tilde{U} \times \Omega \rightarrow D(\omega)$ for a single

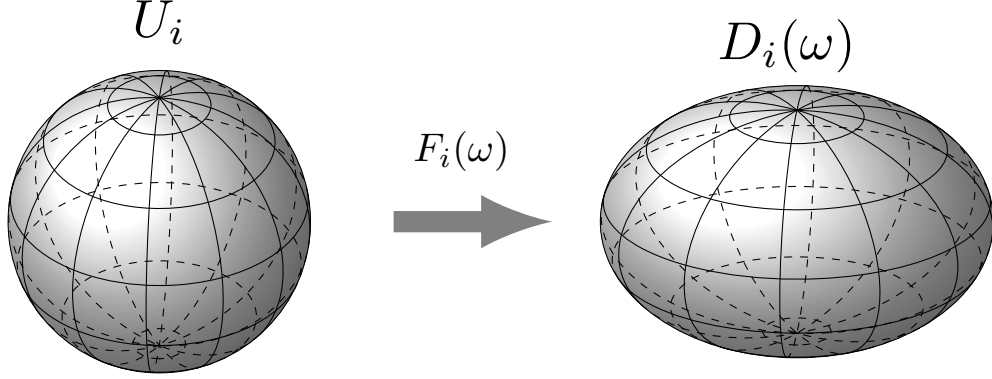


FIGURE 1. Cartoon example of stochastic domain realization from a reference domain.

generic element \tilde{U} with the following parameterization:

$$e(x, \omega) := \sum_{l=1}^N \sqrt{\lambda_l} b_l(x) Y_l(\omega),$$

where $\Gamma_n \equiv Y_n(\Omega)$, $\mathbb{E}[Y_n] = 0$, $\mathbb{E}[Y_n^2] = 1$ and $\Gamma := \prod_{n=1}^N \Gamma_n$.

Assumption 4.

- (1) $n = 1, \dots, N$, let $\Gamma_n \equiv [-1, 1]$
- (2) $b_1, \dots, b_N \in C^\infty(\tilde{U})$
- (3) $\|b_l \sqrt{\lambda_l}\|_{L^\infty(\tilde{U})}$ are monotonically decreasing for $l = 1, 2, \dots, N$.

3. ANALYTICITY

The analysis in this section is based on the analytical extension derived in Section 3 of [4] for the elliptic PDE case but extended to parabolic PDEs with a random diffusion coefficient. Without loss of generality assume that $F : \tilde{U} \times \Omega \rightarrow \mathbb{R}^d$ is parameterized by one single element \tilde{U} . From Assumption 3 the Jacobian can be written as

$$(6) \quad \partial F(x, \omega) = I + \sum_{l=1}^N B_l(x) \sqrt{\lambda_l} Y_l(\omega)$$

$$\text{where } B_l(x) := b_l(x) \partial \hat{v}(x) + \begin{bmatrix} \frac{\partial b_l(x)}{\partial x_1} \hat{v}_1(x) & \frac{\partial b_l(x)}{\partial x_2} \hat{v}_1(x) & \dots & \frac{\partial b_l(x)}{\partial x_d} \hat{v}_1(x) \\ \frac{\partial b_l(x)}{\partial x_1} \hat{v}_2(x) & \frac{\partial b_l(x)}{\partial x_2} \hat{v}_2(x) & \dots & \frac{\partial b_l(x)}{\partial x_d} \hat{v}_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial b_l(x)}{\partial x_1} \hat{v}_d(x) & \frac{\partial b_l(x)}{\partial x_2} \hat{v}_d(x) & \dots & \frac{\partial b_l(x)}{\partial x_d} \hat{v}_d(x) \end{bmatrix} \text{ and } \partial \hat{v} \text{ is the Jaco-}$$

bian of $\hat{v}(x)$.

Assumption 5.

- (a) There exists $0 < \tilde{\delta} < 1$ such that $\sum_{l=1}^N \|B_l(x)\|_2 \sqrt{\lambda_l} \leq 1 - \tilde{\delta}$, $\forall x \in \tilde{U}$ and $\sum_{l=1}^N \|b_l(x)\|_\infty \sqrt{\lambda_l} \leq 1 - \tilde{\delta}$, $\forall x \in \tilde{U}$.

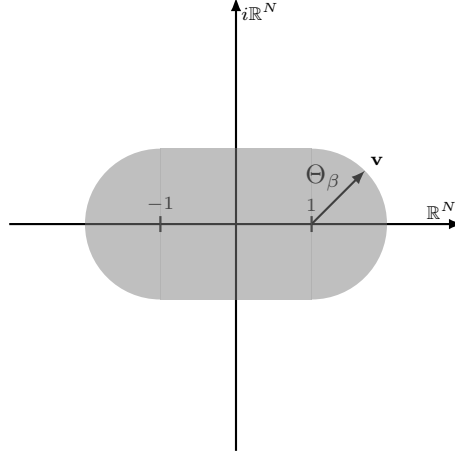


FIGURE 2. Analytic extension of $\tilde{u} : \Gamma \rightarrow L^2(0, T; V)$ to $\tilde{u} : \Theta_\beta \subset \mathbb{C}^N \rightarrow L^2(0, T; V)$.

(b) Assume that $f : U \rightarrow L^2(U)$, $\mathbf{w} : U \rightarrow L^2(U)$ and $g_2 : U_2 \rightarrow H^{1/2}(U)$ can be analytically extended in \mathbb{C}^d .

(c) Assume that $a : D(\omega) \rightarrow \mathbb{R}$ satisfies $\sum_{k=1}^{\infty} \mathcal{Q}_N^k \sum_{1 \leq |\boldsymbol{\lambda}| \leq k} b(k, \boldsymbol{\lambda}) |\partial_{F_1}^{\lambda_1} \dots \partial_{F_d}^{\lambda_d} a| < \frac{a_{\min}}{1+c}$ a.s.

where $c := 1/\tan(\pi/8)$, $\mathcal{Q}_N := \sum_{l=1}^N \sqrt{\lambda_l} \|b_l\|_{L^\infty(U)}$, $b(k, \boldsymbol{\lambda}) := \sum_{p(k, \boldsymbol{\lambda})} \prod_{i=1}^d \frac{v_i^{k_i}}{k_i!}$ and $p(k, \boldsymbol{\lambda}) := \{\mathbf{k}_1, \dots, \mathbf{k}_k \in \mathbb{N}^d : \forall i = 1, \dots, k \text{ } |\mathbf{k}_i| \geq 0, \sum_{j=1}^k \mathbf{k}_j = \boldsymbol{\lambda} \text{ and } \sum_{j=1}^k |\mathbf{k}_j| = k\}$.

Remark 1. In general the absolute value of a complex function is not holomorphic in \mathbb{C} unless it is a constant. Thus a necessary condition for the Neumann boundary condition term in $\tilde{l}(\omega; v)$, $\forall v \in V$, to be holomorphic in Problem 3 is that the real part of $|\partial F(\omega)^{-T} \nabla S_{\partial U}|$ is a constant or $|\partial F(\omega)^{-T} \nabla S_{\partial U}| = |\nabla S_{\partial U}|$.

Now, rewrite the mapping as $\partial F(\mathbf{y}) = I + R(\mathbf{y})$ where $R(\mathbf{y}) := \sum_{l=1}^N \sqrt{\lambda_l} B_l(x) y_l$. This map can be extended into the complex plane. For any $0 < \beta < \tilde{\delta}$ define the following region in \mathbb{C}^N :

$$(7) \quad \Theta_\beta := \left\{ \mathbf{z} \in \mathbb{C}^N; \mathbf{z} = \mathbf{y} + \mathbf{v}, \mathbf{y} \in [-1, 1]^N, \sum_{l=1}^N \sup_{x \in U} \|B_l(x)\|_2 \sqrt{\lambda_l} |v_l| \leq \beta \right\}.$$

The main objective is now to find an extension of the solution \tilde{u} into the region defined by Θ_β as shown in Figure 2.

Lemma 2. Let $0 < \beta < \tilde{\delta}$ then for all $\mathbf{z} \in \Theta_\beta$ we have that i) $a \circ F$ is holomorphic, ii) $\operatorname{Re} a \circ F \geq \frac{a_{\min c}}{1+c}$, iii) $|\operatorname{Im} a \circ F| < \frac{a_{\min}}{1+c}$ and iv) $|a \circ F| \leq \frac{(1+c)a_{\max} + a_{\min}}{1+c}$.

Proof. Let $z := \sum_{l=1}^N \sqrt{\lambda_l} b_l(x) z_l$, then from Corollary 2.11 in [5]

$$\partial_z^k a \circ F = \sum_{1 \leq |\boldsymbol{\lambda}| \leq k} \tilde{b}(k, \boldsymbol{\lambda}) \partial_{F_1}^{\lambda_1} \dots \partial_{F_d}^{\lambda_d} a$$

where $\tilde{b}(k, \boldsymbol{\lambda}) := k! \sum_{p(k, \boldsymbol{\lambda})} \prod_{j=1}^k \prod_{i=1}^d \frac{(\partial_z^j F_i)^{k_j^i}}{k_j^i! j!^{k_j^i}}$. Now observe that $\partial_z^j F_i = 0$ for $j = 2, \dots$ and $i = 1, \dots, d$ then $\tilde{b}(k, \boldsymbol{\lambda}) = k! \sum_{p(k, \boldsymbol{\lambda})} \prod_{i=1}^d \frac{\tilde{v}_1^{k_1^i}}{k_1^i!}$. Now,

$$\sum_{k=1}^{\infty} \frac{|\partial_z^k a(F_1, \dots, F_d)|}{k!} |z|^k \leq \sum_{k=1}^{\infty} \sum_{1 \leq |\boldsymbol{\lambda}| \leq k} b(k, \boldsymbol{\lambda}) |\partial_{F_1}^{\lambda_1} \dots \partial_{F_d}^{\lambda_d} a| |z|^k < \frac{a_{\min}}{1+c}.$$

From Taylor's Theorem it follows that $a \circ F$ is holomorphic for all $\beta \in \Theta_\beta$. It also follows *ii*), *iii*) and *iv*).

□

Lemma 3. *Let $G(\mathbf{z}) := (a \circ F) \det(\partial F(\mathbf{z})) \partial F^{-1}(\mathbf{z}) \partial F^{-T}(\mathbf{z})$ and suppose*

$$0 < \beta < \min\left\{\tilde{\delta} \frac{\log(2-\gamma)}{d + \log(2-\gamma)}, \sqrt{1 + \tilde{\delta}^2/2} - 1\right\}$$

where $\gamma := \frac{c(2-\tilde{\delta})^d}{\tilde{\delta}^d + c(2-\tilde{\delta})^d}$ then $\operatorname{Re} G(\mathbf{z})$ is positive definite $\forall \mathbf{z} \in \Theta_\beta$ and

(a) $\lambda_{\min}(\operatorname{Re} G(\mathbf{z})^{-1}) \geq \tilde{B}(\tilde{\delta}, \beta, d, a_{\max}) > 0$ where

$$\tilde{B}(\tilde{\delta}, \beta, d, a_{\max}, a_{\min}) := \frac{a_{\min}(c^2 - 1)\tilde{\delta}(\tilde{\delta} - 2\beta)(\tilde{\delta}^d \alpha - (2 - \tilde{\delta})^d(2 - \alpha))}{((1+c)a_{\max} + a_{\min})^2(2 - \tilde{\delta})^{2d}(2 - \alpha)^2}.$$

(b) $\lambda_{\max}(\operatorname{Re} G(\mathbf{z})^{-1}) \leq \tilde{D}(\tilde{\delta}, \beta, d, a_{\min}) < \infty$ where

$$\tilde{D}(\tilde{\delta}, \beta, d, a_{\min}) := \frac{1+c}{a_{\min} c \tilde{\delta}^d \alpha} \left[(2 - \tilde{\delta} + \beta)^2 + 2\beta(2 + (\beta - \tilde{\delta})) \right].$$

(c) $\sigma_{\max}(\operatorname{Im} G(\mathbf{z})^{-1}) \leq \tilde{C}(\tilde{\delta}, \beta, d, a_{\min}) < \infty$ where

$$\tilde{C}(\tilde{\delta}, \beta, d, a_{\min}) := \frac{(1+c)(2\beta - \tilde{\delta} + 2)^2}{a_{\min} c \tilde{\delta}^d \alpha}.$$

Proof. (a) To simplify the proof the following property is used: if $\operatorname{Re} G^{-1}(\mathbf{z})$ is positive definite then $\operatorname{Re} G(\mathbf{z})$ is positive definite (From (b) in [15]). First, derive bounds for $\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})$ and $\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z})$, thus for all $\mathbf{z} \in \Theta_\beta$

$$\begin{aligned} \operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z}) &= \operatorname{Re}[(I + R(\mathbf{y}) + R(\mathbf{w}))^T (I + R(\mathbf{z}) + R(\mathbf{w}))] \\ &= (I + R(\mathbf{y}) + R_r(\mathbf{w}))^T (I + R(\mathbf{y}) + R_r(\mathbf{w})) - R_i(\mathbf{w})^T R_i(\mathbf{w}). \end{aligned}$$

where $R(\mathbf{w}) = R_r(\mathbf{w}) + iR_i(\mathbf{w})$. By applying the dual Lidskii inequality (if $A, B \in \mathbb{C}^{d \times d}$ are Hermitian then $\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B)$) and thus

$$\begin{aligned} \lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) &\geq \lambda_{\min}((I + R(\mathbf{y}) + R_r(\mathbf{w}))^T (I + R(\mathbf{y}) + R_r(\mathbf{w}))) \\ &\quad - \lambda_{\max}(R_i(\mathbf{w})^T R_i(\mathbf{w})) \\ &= \sigma_{\min}^2(I + R(\mathbf{y}) + R_r(\mathbf{w})) - \sigma_{\max}^2(R_i(\mathbf{w})) \\ &\geq (\sigma_{\min}(I + R(\mathbf{y})) - \sigma_{\max}(R_r(\mathbf{w})))^2 - \sigma_{\max}^2(R_i(\mathbf{w})) \\ &\geq (\tilde{\delta} - \beta)^2 - \beta^2. \end{aligned} \tag{8}$$

It follows that if $\beta < \tilde{\delta}/2$ then $\lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) \geq \tilde{\delta}(\tilde{\delta} - 2\beta) > 0$ and is positive definite. Now, for all $\mathbf{z} \in \Theta_\beta$,

$$(9) \quad \begin{aligned} \max_{i=1,\dots,d} |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))| &\leq \sigma_{\max}(R_i(\mathbf{w})^T(I + R(\mathbf{y}) + R_r(\mathbf{w})) \\ &\quad + (I + R(\mathbf{y}) + R_r(\mathbf{w}))^T R_i(\mathbf{w})) \\ &\leq 2\sigma_{\max}(R_i(\mathbf{w}))\sigma_{\max}(I + R(\mathbf{y}) + R_r(\mathbf{w})) \\ &\leq 2\beta(2 + (\beta - \tilde{\delta})), \end{aligned}$$

thus

$$\begin{aligned} \operatorname{Re} G(\mathbf{z})^{-1} &= \operatorname{Re} \left(\frac{(a_R(\mathbf{z}) - ia_I(\mathbf{z}))}{|a(\mathbf{z})|^2} \frac{(\xi_R(\mathbf{z}) - i\xi_I(\mathbf{z}))}{|\xi(\mathbf{z})|^2} (\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z}) - i \operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) \right) \\ &= \operatorname{Re} \left(\frac{e^{-i\theta_{a(\mathbf{z})}}}{|a(\mathbf{z})|} \frac{e^{-i\theta_{\xi(\mathbf{z})}}}{|\xi(\mathbf{z})|} (\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z}) - i \operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) \right) \end{aligned}$$

where $\xi(\mathbf{z}) := \xi_R(\mathbf{z}) + i\xi_I(\mathbf{z}) = |\xi(\mathbf{z})|e^{i\theta_{\xi(\mathbf{z})}} = \det(I + R(\mathbf{z}))$ and $a(\mathbf{z}) := a_R(\mathbf{z}) + ia_I(\mathbf{z}) = |a(\mathbf{z})|e^{i\theta_{a(\mathbf{z})}}$. It follows that

$$(10) \quad \lambda_{\min}(\operatorname{Re} G(\mathbf{z})^{-1}) \geq \psi_R(\mathbf{z})\lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) - |\psi_I(\mathbf{z})| \max_{i=1,\dots,d} |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))|,$$

where $\psi_R(\mathbf{z}) := \operatorname{Re} a^{-1}(\mathbf{z})\xi^{-1}(\mathbf{z})$ and $\psi_I(\mathbf{z}) := \operatorname{Im} a^{-1}(\mathbf{z})\xi^{-1}(\mathbf{z})$. Furthermore,

I) From Lemma 4 in [4] *iii*) if $\beta < \frac{\tilde{\delta} \log(2-\gamma)}{d+\log(2-\gamma)}$, $\gamma := \frac{c(2-\tilde{\delta})^d}{\tilde{\delta}^d + c(2-\tilde{\delta})^d}$ then $\xi_R(\mathbf{z}) > c|\xi_I(\mathbf{z})| \forall \mathbf{z} \in \Theta_\beta$.

II) From Lemma 2 *ii*) and *iii*) it follows that $a_R > c|a_I|$ if $\mathbf{z} \in \Theta_\beta$.

III) From inequalities (8) and (9) it follows that if $\beta < \sqrt{1 + \tilde{\delta}^2/2} - 1$ then

$$\lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) > \max_{i=1,\dots,d} |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))|.$$

From I) - II) it follows that $\psi_R(\mathbf{z}) > |\psi_I(\mathbf{z})|$ since the angle of $\psi(\mathbf{z})$ is less than $\pi/2$ for all $\mathbf{z} \in \Theta_\beta$. However, an explicit expression can be derived. First,

$$\psi_R(\mathbf{z}) = \frac{a_R(\mathbf{z})\xi_R(\mathbf{z}) + a_I(\mathbf{z})\xi_I(\mathbf{z})}{|a(\mathbf{z})|^2|\xi(\mathbf{z})|^2} \quad \text{and} \quad |\psi_I(\mathbf{z})| \leq |\tilde{\psi}_I(\mathbf{z})| := \frac{a_R(\mathbf{z})|\xi_I(\mathbf{z})| + |a_I(\mathbf{z})|\xi_R(\mathbf{z})}{|a(\mathbf{z})|^2|\xi(\mathbf{z})|^2},$$

thus

$$\begin{aligned} \psi_R(\mathbf{z}) - |\tilde{\psi}_I(\mathbf{z})| &:= \Lambda(\mathbf{z}) \geq \frac{a_R(\mathbf{z})(\xi_R(\mathbf{z}) - |\xi_I(\mathbf{z})|) - |a_I(\mathbf{z})|(\xi_R(\mathbf{z}) - |\xi_I(\mathbf{z})|)}{|a(\mathbf{z})|^2|\xi(\mathbf{z})|^2} \\ &> \frac{a_{\min}(c^2 - 1)(\tilde{\delta}^d \alpha - (2 - \tilde{\delta})^d(2 - \alpha))}{((1 + c)a_{\max} + a_{\min})^2(2 - \tilde{\delta})^{2d}(2 - \alpha)^2} > 0 \end{aligned}$$

Substituting in (10) it follows that for all $\mathbf{z} \in \Theta_\beta$

$$\begin{aligned} \lambda_{\min}(\operatorname{Re} G(\mathbf{z})^{-1}) &\geq \Lambda(\mathbf{z})\lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) \\ &\quad + |\psi_I(\mathbf{z})|(\lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) - \max_{i=1,\dots,d} |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))|) \\ &\geq \Lambda(\mathbf{z})\lambda_{\min}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) \geq \tilde{B}(\tilde{\delta}, \beta, d, a_{\max}) > 0, \end{aligned}$$

where

$$\tilde{B}(\tilde{\delta}, \beta, d, a_{\max}, a_{\min}) := \frac{a_{\min}(c^2 - 1)\tilde{\delta}(\tilde{\delta} - 2\beta)(\tilde{\delta}^d \alpha - (2 - \tilde{\delta})^d(2 - \alpha))}{((1 + c)a_{\max} + a_{\min})^2(2 - \tilde{\delta})^{2d}(2 - \alpha)^2}.$$

From London's Lemma [15] it follows that $\operatorname{Re} G(\mathbf{z})$ is positive definite $\forall \mathbf{z} \in \Theta_\beta$.

(b) By applying the Lidskii inequality (If $A, B \in \mathbb{C}^{d \times d}$ are Hermitian then $\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$) it follows that

$$\begin{aligned}
 \lambda_{\max}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) &\leq \lambda_{\max}((I + R(\mathbf{y}) + R_r(\mathbf{w}))^T (I + R(\mathbf{y}) + R_r(\mathbf{w}))) \\
 &\quad - \lambda_{\min}(R_i(\mathbf{w})^T R_i(\mathbf{w})) \\
 &= \sigma_{\max}^2(I + R(\mathbf{y}) + R_r(\mathbf{w})) - \sigma_{\min}^2(R_i(\mathbf{w})) \\
 &\leq (\sigma_{\max}(I + R(\mathbf{y})) + \sigma_{\max}(R_r(\mathbf{w})))^2 \\
 &\leq (2 - \tilde{\delta} + \beta)^2.
 \end{aligned}
 \tag{11}$$

and

$$\begin{aligned}
 \max_i |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))| &\leq \max_i |\lambda_i(R_i(\mathbf{w})^T (I + R(\mathbf{y}) + R_r(\mathbf{w}))) \\
 &\quad + (I + R(\mathbf{y}) + R_r(\mathbf{w}))^T R_i(\mathbf{w}))| \\
 &\leq 2\beta(2 + (\beta - \tilde{\delta})).
 \end{aligned}
 \tag{12}$$

Now,

$$\begin{aligned}
 \lambda_{\max}(\operatorname{Re} G(\mathbf{z})^{-1}) &\leq \frac{|\psi_R(\mathbf{z})| \lambda_{\max}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) + |\psi_I(\mathbf{z})| \max_i |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))|}{|\psi(\mathbf{z})|^2} \\
 &\leq \frac{\lambda_{\max}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) + \max_i |\lambda_i(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))|}{|\psi(\mathbf{z})|} \\
 &\leq \tilde{D}(\tilde{\delta}, \beta, d, a_{\min}) < \infty.
 \end{aligned}$$

From Lemma 4 in [4] and Lemma 2, it follows that $|\psi(\mathbf{z})|^{-1} = \frac{1}{|a(\mathbf{z})\xi(\mathbf{z})|} \leq \frac{1+c}{a_{\min} c \tilde{\delta}^d \alpha}$ and therefore from inequalities (11) and (12)

$$\tilde{D}(\tilde{\delta}, \beta, d, a_{\min}) := \frac{1+c}{a_{\min} c \tilde{\delta}^d \alpha} \left[(2 - \tilde{\delta} + \beta)^2 + 2\beta(2 + (\beta - \tilde{\delta})) \right].$$

(c) Similarly to (b) it can be shown that

$$\begin{aligned}
 \sigma_{\max}(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) &\leq \sigma_{\max}(R_i(\mathbf{w})^T (I + R(\mathbf{y}) + R_r(\mathbf{w}))) \\
 &\quad + (I + R(\mathbf{y}) + R_r(\mathbf{w}))^T R_i(\mathbf{w})) \\
 &\leq 2\sigma_{\max}(R_i(\mathbf{w}))\sigma_{\max}(I + R(\mathbf{y}) + R_r(\mathbf{w})) \\
 &\leq 2\beta(2 + (\beta - \tilde{\delta})).
 \end{aligned}
 \tag{13}$$

and

$$\begin{aligned}
 \sigma_{\max}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) &\leq \sigma_{\max}((I + R(\mathbf{y}) + R_r(\mathbf{w}))^T (I + R(\mathbf{y}) + R_r(\mathbf{w}))) \\
 &\quad + \sigma_{\max}(R_i(\mathbf{w})^T R_i(\mathbf{w})) \\
 &= \sigma_{\max}^2(I + R(\mathbf{y}) + R_r(\mathbf{w})) + \sigma_{\max}^2(R_i(\mathbf{w})) \\
 &\leq (\sigma_{\max}(I + R(\mathbf{y})) + \sigma_{\max}(R_r(\mathbf{w})))^2 + \sigma_{\max}^2(R_i(\mathbf{w})) \\
 &\leq ((2 - \tilde{\delta}) + \beta)^2 + \beta^2.
 \end{aligned}
 \tag{14}$$

From inequalities (13) and (14), Lemma 3 in [4] and Lemma 4 it follows that

$$\begin{aligned}\sigma_{\max}(\operatorname{Im} G(\mathbf{z})^{-1}) &\leq \frac{|\psi_R(\mathbf{z})|\sigma_{\max}(\operatorname{Im} \partial F(\mathbf{z})^T \partial F(\mathbf{z})) + |\psi_I(\mathbf{z})|\sigma_{\max}(\operatorname{Re} \partial F(\mathbf{z})^T \partial F(\mathbf{z}))}{|\psi(\mathbf{z})|^2} \\ &= \tilde{C}(\tilde{\delta}, \beta, d, a_{\min}) < \infty.\end{aligned}$$

□

Lemma 4. $G(\mathbf{z})$ is positive definite $\forall \mathbf{z} \in \Theta_\beta$ whenever

$$(15) \quad 0 < \beta < \min\left\{\tilde{\delta} \frac{\log(2-\gamma)}{d + \log(2-\gamma)}, \sqrt{1 + \tilde{\delta}^2/2} - 1\right\}$$

where $\gamma := \frac{c(2-\tilde{\delta})^d}{\tilde{\delta}^d + c(2-\tilde{\delta})^d}$. Furthermore $\lambda_{\min}(\operatorname{Re} G(\mathbf{z})) \geq \varepsilon(\tilde{\delta}, \beta, d, a_{\max}, a_{\min}) > 0$ where

$$\varepsilon(\tilde{\delta}, \beta, d, a_{\max}, a_{\min}) := \frac{1}{\left(1 + \left(\frac{\tilde{C}(\tilde{\delta}, \beta, d, a_{\min})}{\tilde{B}(\tilde{\delta}, \beta, d, a_{\max})}\right)^2\right) \tilde{D}(\tilde{\delta}, \beta, d, a_{\min})}.$$

Proof. The proof essentially follows Lemma 6 in [4].

□

The main result of this section can now be proven. For $n = 1, \dots, N$ consider the map $\Psi(s) : \Gamma_n \rightarrow L^2(0, T; V(U))$ where $\Psi(s) := \tilde{u}(y_n(s), \hat{\mathbf{y}}_n, x)$, for any arbitrary point $\hat{\mathbf{y}}_n \in \hat{\Gamma}_n$ where $\hat{\Gamma}_n := \left(\prod_{l=1, \dots, N, l \neq n} \Gamma_l\right)$. Consider the extension of s into the complex plane as $z = s + iw$ in the region Θ_β along the n^{th} dimension. Now, for notational simplicity reorder (y_1, \dots, y_N) such that $n = N$ and extend $\hat{\mathbf{y}}_n \rightarrow \hat{\mathbf{z}} \in \hat{\Theta}_\beta^n$, where $\hat{\Theta}_\beta^n := \Theta_\beta \cap \mathbb{C}^{N-1}$, then $\Psi(s)$ has a natural extension to the complex plane as $\Psi(z) := \tilde{u}(z, \hat{\mathbf{z}}, x)$ for all $\hat{\mathbf{z}} \in \hat{\Theta}_\beta^n \subset \Theta_\beta$.

Theorem 1. Let $0 < \tilde{\delta} < 1$ then $\tilde{u}(\mathbf{z})$ is holomorphic in Θ_β (7) if $\beta < \min\left\{\tilde{\delta} \frac{\log(2-\gamma)}{d + \log(2-\gamma)}, \sqrt{1 + \tilde{\delta}^2/2} - 1\right\}$ where $\gamma := \frac{c(2-\tilde{\delta})^d}{\tilde{\delta}^d + c(2-\tilde{\delta})^d}$.

Proof. This proof closely follows Theorem 1 in [4] and is extended to the parabolic case.

Since $\beta < \tilde{\delta}$ then it is not hard to see that $\partial F^{-1}(\mathbf{z}) = (I + R(\mathbf{z}))^{-1} = I + \sum_{k=1}^{\infty} R(\mathbf{z})^k$ is convergent $\forall \mathbf{z} \in \Theta_\beta$. From Lemma 4 in [4] it follows that $G(\mathbf{z})$ is analytic for all $\mathbf{z} \in \Theta_\beta$.

Now, let $\Psi = [\Psi_R, \Psi_I]^T$, where $\Psi_R = \operatorname{Re} \Psi(z)$ and $\Psi_I = \operatorname{Im} \Psi(z)$. Then Ψ solves in the weak sense the problem

$$(16) \quad \partial_t \Psi(t) - \nabla \cdot \hat{G} \nabla \Psi = \hat{f},$$

where $\hat{G} := \begin{pmatrix} G_R & -G_I \\ G_I & G_R \end{pmatrix}$, $\hat{f} := \begin{pmatrix} f_R \\ f_I \end{pmatrix}$, $G_R := \operatorname{Re}(G)$, $G_I := \operatorname{Im}(G)$, $f_R := \operatorname{Re} \tilde{f}$ and $f_I = \operatorname{Im} \tilde{f}$. Note that \tilde{f} refers to rhs of the weak formulation i.e. $l(z; v)$ for all $v \in V(\tilde{U})$. From Lemma (3) it follows that G_R is positive definite if $\mathbf{z} \in \Theta_\beta$, thus (16) has a unique solution.

To show that $\Psi(z) : \mathbb{C} \rightarrow L^2(0, T; V(\tilde{U}))$ is holomorphic in \mathbb{C} for $n = 1, \dots, N$ the strategy is to show that the Cauchy-Riemann conditions are satisfied. The first step is to show that the

derivatives $\partial_s \Psi$ and $\partial_w \Psi$ exist. Now, differentiating (16) with respect to $s = \operatorname{Re} z$ and $w = \operatorname{Im} z$ one obtains that

$$\begin{aligned}
 (17) \quad & \partial_t \partial_s \Psi(t) - (\nabla \cdot G_R \nabla \partial_s \Psi_R(z) - \nabla \cdot G_I \nabla \partial_s \Psi_I(z)) \\
 & = \nabla \cdot \partial_s G_R \nabla \Psi_R(z) - \nabla \cdot \partial_s G_I \nabla \Psi_I(z) + \partial_s f_R(z), \\
 & \partial_t \partial_s \Psi(t) - (\nabla \cdot G_I \nabla \partial_s \Psi_R(z) + \nabla \cdot G_R \nabla \partial_s \Psi_I(z)) \\
 & = \nabla \cdot \partial_s G_I \nabla \Psi_R(z) + \nabla \cdot \partial_s G_R \nabla \Psi_I(z) + \partial_s f_I(z), \\
 & \partial_t \partial_w \Psi(t) - (\nabla \cdot G_R \nabla \partial_w \Psi_R(z) - \nabla \cdot G_I \nabla \partial_w \Psi_I(z)) \\
 & = \nabla \cdot \partial_w G_R \nabla \Psi_R(z) - \nabla \cdot \partial_w G_I \nabla \Psi_I(z) + \partial_w f_R(z), \\
 & \partial_t \partial_w \Psi(t) - (\nabla \cdot G_I \nabla \partial_w \Psi_R(z) + \nabla \cdot G_R \nabla \partial_w \Psi_I(z)) \\
 & = \nabla \cdot \partial_w G_I \nabla \Psi_R(z) + \nabla \cdot \partial_w G_R \nabla \Psi_I(z) + \partial_w f_I(z).
 \end{aligned}$$

Since G_R is a positive definite for all $\mathbf{z} \in \Theta_\beta$ then the derivatives $\partial_s \Psi$ and $\partial_w \Psi$ exist and are unique.

Now, let $P(z) := \partial_s \Psi_R(z) - \partial_w \Psi_I(z)$ and $Q(z) := \partial_w \Psi_R(z) + \partial_s \Psi_I(z)$. By taking linear combinations of eqns (17):

$$\begin{aligned}
 (18) \quad & \partial_t P(z) - \nabla \cdot (G_R \nabla P - G_I \nabla Q) = \nabla \cdot ((\partial_s G_R - \partial_w G_I) \nabla \Psi_R - (\partial_w G_R + \partial_s G_I) \nabla \Psi_I \\
 & \quad + \partial_s f_R - \partial_w f_I) \\
 & \partial_t Q(z) - \nabla \cdot (G_I \nabla P + G_R \nabla Q) = \nabla \cdot ((\partial_w G_R + \partial_s G_I) \nabla \Psi_R - (\partial_s G_R - \partial_w G_I) \nabla \Psi_I \\
 & \quad + \partial_s f_I + \partial_w f_R)
 \end{aligned}$$

The following step is to shown that $G(z)$ and $\hat{f}(z)$ satisfies the Riemann-Cauchy conditions so that the right hand side becomes zero. From Assumption 5 (b) it follows that $f \circ F(z)$ and $\mathbf{w} \circ F$ can be analytically extended in \mathbb{C} . Furthermore, from Remark 1 and Assumption 5 it follows that that $g(z, \hat{\mathbf{z}}) := \frac{\det(\partial F(z)) |\partial F^{-T}(z) \nabla S_{\partial U}| g_2 \circ F(z)}{|\nabla S_{\partial U}|}$ can be analytically extended in \mathbb{C} . Since $\sum_{k=0}^{\infty} g(x, z_n, \hat{\mathbf{z}}_n) z^k$ is absolutely convergent in $L^2(\partial U)$ then $\int_{U_2} g(z_n, \hat{\mathbf{z}})$ is analytic in $z \in \Theta_\beta$ for all $\hat{\mathbf{z}} \in \hat{\Theta}_\beta^n$. Now, recall that $G(\mathbf{z})$ is analytic if $\mathbf{z} \in \Theta_\beta$ and it follows that $l(z, \hat{\mathbf{z}}; v)$ is holomorphic for all $z \in \Theta_\beta$, $\hat{\mathbf{z}} \in \hat{\Theta}_\beta^n$, and $v \in V$. Thus equations (18) have a unique solution $P(z) = Q(z) = 0$ for all $z \in \Theta_\beta$ and $\hat{\mathbf{z}} \in \hat{\Theta}_\beta^n$. From the Looman-Menchoff theorem $\Psi(z)$ is holomorphic for all $z \in \Theta_\beta$ and $\hat{\mathbf{z}} \in \hat{\Theta}_\beta^n$.

The next step is to extend the analyticity of the solution $\tilde{u}(\mathbf{z})$ to the entire domain Θ_β . Repeat the analytic extension of $\tilde{u}(y_n, \hat{\mathbf{y}}_n, x)$ for $n = 1, \dots, N$. Since each variable $\tilde{u}(y_n, \hat{\mathbf{y}}_n, x)$ has been extended into the complex plane for $z \in \Theta_\beta$ and $\hat{\mathbf{z}} \in \hat{\Theta}_\beta^n$ from Hartog's Theorem it follows that $\Psi(\mathbf{z})$ is continuous in Θ_β . From Osgood's Lemma it follows that $\Psi(\mathbf{z})$ is holomorphic for all $\mathbf{z} \in \Theta_\beta$. \square

4. STOCHASTIC COLLOCATION

For sake of completeness and to be consistent with the same notation of [4], a modification of section 4 from [4] is introduced. The goal of this paper efficiently compute a numerical approximation to the exact moments of the QoI of the form (4) in a finite dimensional subspace based on a tensor product structure is sought.

Suppose $\mathcal{P}_p(\Gamma) \subset L^2_\rho(\Gamma)$ is the span of tensor product polynomials of degree at most $p = (p_1, \dots, p_N)$; i.e., $\mathcal{P}_p(\Gamma) = \bigotimes_{n=1}^N \mathcal{P}_{p_n}(\Gamma_n)$ with $\mathcal{P}_{p_n}(\Gamma_n) = \text{span}(y_n^m, m = 0, \dots, p_n), n = 1, \dots, N$. It is easy to see that the dimension of \mathcal{P}_p is $N_p = \prod_{n=1}^N (p_n + 1)$.

In many cases it is not feasible to directly compute the statistical moments for a high dimensional $\rho(\mathbf{y})$. Alternatively, an auxiliary probability density function $\hat{\rho} : \Gamma \rightarrow \mathbb{R}^+$ can be used such that it is formed from the joint probability of N independent random variables:

$$(19) \quad \hat{\rho}(\mathbf{y}) = \prod_{n=1}^N \hat{\rho}_n(y_n) \quad \forall \mathbf{y} \in \Gamma, \quad \text{and is such that} \quad \left\| \frac{\rho}{\hat{\rho}} \right\|_{L^\infty(\Gamma)} < C_\tau$$

for some bounded constant $C_\tau > 0$.

The next step consists in collocating $Q(\tilde{u}(\mathbf{y}))$ with respect to Γ . For each dimension $n = 1, \dots, N$, let $y_{n,k_n}, 1 \leq k_n \leq p_n + 1$, be the $p_n + 1$ roots of the orthogonal polynomial q_{p_n+1} with respect to the weight $\hat{\rho}_n$ i.e. $\int_{\Gamma_n} q_{p_n+1}(\mathbf{y}) v(\mathbf{y}) \hat{\rho}_n(\mathbf{y}) dy = 0$ for all $v \in \mathcal{P}_{p_n}(\Gamma_n)$. Thus, for many choices of $\hat{\rho}$, such as Gaussian, constant, etc, the roots of the polynomial q_{p_n+1} can be pre-computed.

To any vector of indexes $[k_1, \dots, k_N]$ associate the global index $k = k_1 + p_1(k_2 - 1) + p_1 p_2(k_3 - 1) + \dots$ and denote by y_k the point $y_k = [y_{1,k_1}, y_{2,k_2}, \dots, y_{N,k_N}] \in \Gamma$. Furthermore, for each $n = 1, 2, \dots, N$, the Lagrange basis $\{l_{n,j}\}_{j=1}^{p_n+1}$ of the space \mathcal{P}_{p_n} ,

$$l_{n,j} \in \mathcal{P}_{p_n}(\Gamma_n), \quad l_{n,j}(y_{n,k}) = \tilde{\delta}_{jk}, \quad j, k = 1, \dots, p_n + 1,$$

where $\tilde{\delta}_{jk}$ is the Kronecker symbol, and set $l_k(\mathbf{y}) = \prod_{n=1}^N l_{n,k_n}(y_n)$. Now, let $\mathcal{I}_p : C^0(\Gamma) \rightarrow \mathcal{P}_p(\Gamma)$, such that $\mathcal{I}_p v(\mathbf{y}) = \sum_{k=1}^{N_p} v(y_k) l_k(\mathbf{y}) \forall v \in C^0(\Gamma)$. Thus for any $\mathbf{y} \in \Gamma$ the Lagrange approximation of the QoI ($Q(\mathbf{y})$) can be written as: $Q_p(u) := \mathcal{I}_p \int_\Gamma \tilde{u}(\mathbf{y}) \partial_t \varphi(\mathbf{y}) + B(\mathbf{y}; \tilde{u}(\mathbf{y}), \varphi(\mathbf{y}))$.

Now, for any continuous function $g : \Gamma \rightarrow \mathbb{R}$ the integral $\int_\Gamma g(\mathbf{y}) \hat{\rho}(\mathbf{y}) d\mathbf{y}$ can be approximated as by a Gauss quadrature formula $\mathbb{E}_\rho^p[g]$ as

$$(20) \quad \mathbb{E}_\rho^p[g] = \sum_{k=1}^{N_p} \omega_k g(y_k), \quad \omega_k = \prod_{n=1}^N \omega_{k_n}, \quad \omega_{k_n} = \int_{\Gamma_n} l_{k_n}^2(y) \hat{\rho}_n(y) dy.$$

However, without loss of generality, the quadrature scheme for the expectation is assumed to be exact.

4.1. Sparse Grid Approximation. From section 4 it is shown that the dimension of \mathcal{P}_p increases as $\prod_{n=1}^N (p_n + 1)$ i.e. the number of PDE solves grows exponentially with respect to N , which is impractical. If the stochastic integral regularity is high, this problem can be mitigate with the application of a Smolyak sparse grid. The basic notation and construction of Smolyak sparse grids are presented in this section. The reader is encouraged to read [18, 2, 1] for more details.

Let $\mathcal{I}_n^{m(i)} : C^0(\Gamma_n) \rightarrow \mathcal{P}_{m(i)-1}(\Gamma_n)$ where $i \geq 1$ denotes the level of approximation and $m(i)$ the number of collocation points used to build the interpolation at level i . Now, set $m(1) = 1$ and $m(i) < m(i+1)$ for $i \geq 1$. Furthermore, let $m(0) = 0$ and $\mathcal{I}_n^{m(0)} = 0$ and $\Delta_n^{m(i)} := \mathcal{I}_n^{m(i)} - \mathcal{I}_n^{m(i-1)}$.

Given the approximation level integer $w \geq 0$, and the multi-index $\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{N}_+^N$, let $g : \mathbb{N}_+^N \rightarrow \mathbb{N}$ be a strictly increasing function in each argument, the sparse grid approximation of

Q_h is defined as

$$(21) \quad \mathcal{S}_w^{m,g}[Q_h] = \sum_{\mathbf{i} \in \mathbb{N}_+^N : g(\mathbf{i}) \leq w} \bigotimes_{n=1}^N \Delta_n^{m(i_n)}(Q_h).$$

Notice that in contrast to the full tensor grid given by \mathcal{I}_p the constraint $g(\mathbf{i}) \leq w$ in (21) is chosen so as to limit the use of tensor grids of high polynomial degree across all dimensions.

Let $\mathbf{m}(\mathbf{i}) = (m(i_1), \dots, m(i_N))$ and $\Lambda^{m,g}(w) = \{\mathbf{p} \in \mathbb{N}^N, \ g(\mathbf{m}^{-1}(\mathbf{p} + \mathbf{1})) \leq w\}$. Denote by $\mathbb{P}_{\Lambda^{m,g}(w)}(\Gamma) = \text{span} \left\{ \prod_{n=1}^N y_n^{p_n}, \text{ with } \mathbf{p} \in \Lambda^{m,g}(w) \right\}$ the corresponding multivariate polynomial space spanned by the monomials with multi-degree in $\Lambda^{m,g}(w)$. The most typical choice of m and g that leads to the Smolyak (SM) sparse grid is given by (see [18, 2])

$$m(i) = \begin{cases} 2^{i-1} + 1, & i > 1 \\ 1, & i = 1 \end{cases} \quad \Lambda(w) := \{\mathbf{p} \in \mathbb{N}^N : \sum_n f(p_n) \leq w\}$$

$$g(\mathbf{i}) = \sum_n (i_n - 1) \leq w \quad f(p) = \begin{cases} 0, & p = 0 \\ 1, & p = 1 \\ \lceil \log_2(p) \rceil, & p \geq 2. \end{cases}$$

Other choices of $g(\mathbf{i})$, m and $\Lambda^{m,g}$ include the Tensor Product (TP), the Total Degree (TD) and the Hyperbolic Cross (HC) (see [1]).

The next step is to choose the interpolating sparse grid points. The most typical choice of the Clenshaw-Curtis (CC) interpolation points (extrema of Chebyshev polynomials) leads to nested sequences of one dimensional interpolation formulas. This choice leads to a significantly smaller number of interpolating points compared to the corresponding tensor grid. Other choices exists and the reader is referred to [17].

The mean term $\mathbb{E}[Q]$ can now be approximated as

$$(22) \quad \mathbb{E}[\mathcal{S}_w^{m,g}Q] = \mathbb{E}_{\hat{\rho}}[\mathcal{S}_w^{m,g}Q \frac{\rho}{\hat{\rho}}].$$

Furthermore, the variance $\text{var}[Q]$ is approximated as

$$(23) \quad \text{var}[Q] = \mathbb{E}[(\mathcal{S}_w^{m,g}[Q])^2] - \mathbb{E}[\mathcal{S}_w^{m,g}[Q]]^2 = \mathbb{E}_{\hat{\rho}}[(\mathcal{S}_w^{m,g}[Q])^2 \frac{\rho}{\hat{\rho}}] - \mathbb{E}_{\hat{\rho}}[\mathcal{S}_w^{m,g}[Q] \frac{\rho}{\hat{\rho}}]^2.$$

5. ERROR ANALYSIS

In this section error estimates of the mean and variance are derived with respect to the i) sparse grid approximation, ii) the truncation of the stochastic model to the first N_s dimensions and the iii) deterministic solver. However, the error contribution from the deterministic solver is ignored since there are many methods that can be used to solve the parabolic equation (e.g. [13]). Now, for notational simplicity split the Jacobian as follows

$$(24) \quad \partial F(x, \omega) = I + \sum_{l=1}^{N_s} B_l(x) \sqrt{\lambda_l} Y_l(\omega) + \sum_{l=N_s+1}^N B_l(x) \sqrt{\lambda_l} Y_l(\omega).$$

Furthermore, let $\Gamma_s := [-1, 1]^{N_s}$, $\Gamma_f := [-1, 1]^{N-N_s}$, then the domain $\Gamma = \Gamma_s \times \Gamma_f$. Next, refer to $Q(\mathbf{y}_s)$ as $Q(\mathbf{y})$ restricted to the stochastic domain Γ_s and similarly for $G(\mathbf{y}_s)$. It is clear also that $Q(\mathbf{y}_s, \mathbf{y}_f) = Q(\mathbf{y})$ and $G(\mathbf{y}_s, \mathbf{y}_f) = G(\mathbf{y})$ for all $\mathbf{y} \in \Gamma_s \times \Gamma_f$, $\mathbf{y}_s \in \Gamma_s$, and $\mathbf{y}_f \in \Gamma_f$.

The next step is to derive estimates for the variance ($|var[Q(\mathbf{y}_s, \mathbf{y}_f)] - var[\mathcal{S}_w^{m,g}[Q(\mathbf{y}_s)]]|$) and mean ($|\mathbb{E}[Q(\mathbf{y}_s, \mathbf{y}_f)] - \mathbb{E}[\mathcal{S}_w^{m,g}[Q(\mathbf{y}_s)]]|$) errors. It is not hard to show that $|var[Q(\mathbf{y}_s, \mathbf{y}_f)] - var[\mathcal{S}_w^{m,g}[Q_h(\mathbf{y}_s)]]|$ and $|\mathbb{E}[Q(\mathbf{y}_s, \mathbf{y}_f)] - \mathbb{E}[\mathcal{S}_w^{m,g}[Q_h(\mathbf{y}_s)]]|$ are less or equal to (see [4])

$$\underbrace{C_T \|Q(\mathbf{y}_s, \mathbf{y}_f) - Q(\mathbf{y}_s)\|_{L_P^2(\Gamma)}}_{\text{Truncation (I)}} + \underbrace{C_{SG} \|Q(\mathbf{y}_s) - \mathcal{S}_w^{m,g}[Q(\mathbf{y}_s)]\|_{L_P^2(\Gamma_s)}}_{\text{Sparse Grid (II)}},$$

for some positive constants C_T and C_{SG} .

5.1. Truncation Error (I). Given that $Q : V \rightarrow \mathbb{R}$ is a bounded linear functional then for any realization of $\varphi(\mathbf{y}_s, \mathbf{y}_f)$ it follows that

$$\begin{aligned} |Q(\mathbf{y}_s, \mathbf{y}_f) - Q(\mathbf{y}_s)| &= \left| \int_U (\tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)) \partial_t \varphi \right. \\ &\quad \left. + B(\mathbf{y}_s, \mathbf{y}_f; \varphi(\mathbf{y}_s, \mathbf{y}_f), \tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)) \right| \\ &\leq a_{max} \mathbb{F}_{max}^d \mathbb{F}_{min}^{-2} \|\varphi(\mathbf{y}_s, \mathbf{y}_f)\|_V \|\tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)\|_V \\ &\quad + C_P(U) \|\partial_t \varphi(\mathbf{y}_s, \mathbf{y}_f)\|_{L^2(U)} \|\tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)\|_V. \end{aligned}$$

where $C_P(U)$ is the Poincaré constant. Note that with a slight abuse of notation refer to $\tilde{u} \circ F(\mathbf{y})$ as $\tilde{u}(\mathbf{y})$. Now, from Theorem 5 [6] it follows that

$$\|\partial_t \varphi(\mathbf{y}_s, \mathbf{y}_f)\|_{L^2(U)} \leq Ct \|q\|_{L^2(U)} \quad \text{and} \quad \|\varphi(\mathbf{y}_s, \mathbf{y}_f)\|_V \leq Ct \|q\|_{L^2(U)}$$

for some $C > 0$. It follows that for $T \geq t > 0$

$$\|Q(\mathbf{y}_s, \mathbf{y}_f) - Q(\mathbf{y}_s)\|_{L_P^2(\Gamma)} \leq C_{TR}(t) \|\tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)\|_{L_P^2(\Gamma; V)}$$

where $C_{TR}(t) := Ct \|q\|_{L^2(U)} (a_{max} \mathbb{F}_{max}^d \mathbb{F}_{min}^{-2} + 1)$. The objective now is to control the error term $e := \|\tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)\|_{L_P^2(\Gamma; V)}$.

Assumption 6.

- Without loss of generality assume that the Neumann boundary condition is homogeneous i.e. $g_2 = 0$ on U_2 .
- $\forall \mathbf{y} \in \Gamma$ assume that $u_0(\mathbf{y}) \in H^2(U)$, $f(\mathbf{y}) \in L^2(0, T; L_\rho^2(\Gamma) \otimes L^2(U))$, $\partial_t f(\mathbf{y}) \in L^2(0, T; L_\rho^2(\Gamma) \otimes L^2(U))$ and $\mathbf{w}(\mathbf{y}) \in H^2(U)$.

Furthermore, if W is a Banach space defined on $U \times [0, T]$ then let

$$C^0(\Gamma; W) := \{v : \Gamma \rightarrow W \text{ is continuous on } \Gamma \text{ and } \max_{y \in \Gamma} \|v(y)\|_W < \infty\}.$$

and $L_\rho^2(\Gamma; W) := \{v : \Gamma \rightarrow W \text{ is strongly measurable and } \int_\Gamma \|v\|_W^2 \rho(\mathbf{y}) d\mathbf{y} < \infty\}$. From Theorem 1 it follows that $\tilde{u} \in C^0(\Gamma; L^2(0, T; V)) \subset L_\rho^2(\Gamma; L^2(0, T; V))$. Moreover

$$L_\rho^2(\Gamma; L^2(0, T; V)) \cong L_\rho^2(\Gamma) \otimes L^2(0, T; V) \cong L^2(0, T; L_\rho^2(\Gamma) \otimes V)$$

and is endowed with the inner product $(v_1, v_2)_{L^2(0, T; L_\rho^2(\Gamma) \otimes V)} \equiv \int_{[0, T] \times U} \mathbb{E}[\nabla v_1 \cdot \nabla v_2] dx dt$, thus $\tilde{u} \in L^2(0, T; L_\rho^2(\Gamma) \otimes V)$ satisfies the following variational problem a.e.

$$(25) \quad \int_U \mathbb{E}[(\partial_t \tilde{u})v] + \mathcal{A}(\mathbf{y}_s, \mathbf{y}_f; \tilde{u}, v) = \mathbb{E}[\tilde{l}(\mathbf{y}_s, \mathbf{y}_f; v)] \quad \forall v \in L^2(0, T; L_\rho^2(\Gamma) \otimes V)$$

where $\mathcal{A}(\mathbf{y}_s, \mathbf{y}_f; \tilde{u}, v) := \mathbb{E}[B(\mathbf{y}_s, \mathbf{y}_f; \tilde{u}, v)]$. Moreover, the following energy estimates hold from Theorem 5 in [6]:

$$\begin{aligned} \|\tilde{u}(\mathbf{y}, T)\|_{L_\rho^2(\Gamma) \otimes V}^2 + \|\tilde{u}(\mathbf{y})\|_{L^2(0, T; L_\rho^2(\Gamma) \otimes V)}^2 &\leq C(\|f(\mathbf{y})\|_{L^2(0, T; L_\rho^2(\Gamma) \otimes V)} \\ &+ \|\mathbf{w}(\mathbf{y})\|_{L^2(0, T; L_\rho^2(\Gamma) \otimes H^2(U))} + \|u_0(\mathbf{y})\|_{L_\rho^2(\Gamma) \otimes L^2(U)}) \end{aligned}$$

and

$$(26) \quad \begin{aligned} \|\partial_t \tilde{u}(\mathbf{y})\|_{L^2(0, T; L^2(U) \otimes V)} &\leq C(\|f(\mathbf{y})\|_{H^1(0, T; L_\rho^2(\Gamma) \otimes L^2(U))} + \|\mathbf{w}(\mathbf{y})\|_{L^2(0, T; L_\rho^2(\Gamma) \otimes H^2(U))} \\ &+ \|u_0(\mathbf{y})\|_{L_\rho^2(\Gamma) \otimes H^2(U)}), \end{aligned}$$

for some constant $C > 0$.

Theorem 2. *Let $\tilde{u}(\mathbf{y}_s, \mathbf{y}_f)$ be the solution to the bilinear Problem 3 that satisfies Assumptions 1, 2, 3, 4, 5 and 6. Furthermore, let $B_{\mathbb{T}} := \sup_{x \in U} \sum_{i=N_s+1}^N \sqrt{\lambda_i} \|B_i(x)\|$ and $e(\mathbf{y}) := \tilde{u}(\mathbf{y}_s, \mathbf{y}_f) - \tilde{u}(\mathbf{y}_s)$ then for $0 < t \leq T$ it follows that*

$$\begin{aligned} \|e(\mathbf{y}, t)\|_{L^2(U) \otimes V} &\leq \mathcal{E}(\mathbb{F}_{max}, \mathbb{F}_{min}, \delta, d, \|a\|_{W^{1, \infty}(U)}, \|\hat{v}\|_{L^\infty(U)}, \|f(\mathbf{y})\|_{H^1(0, T; L_\rho^2(\Gamma) \otimes L^2(U))}, \\ &\|\mathbf{w}(\mathbf{y})\|_{L^2(0, T; L_\rho^2(\Gamma) \otimes H^2(U))}, \|u_0(\mathbf{y})\|_{L_\rho^2(\Gamma) \otimes H^2(U)}, \|u_0(\mathbf{y})\|_{L_\rho^2(\Gamma) \otimes W^{1, \infty}(U)}, \\ &\sup_{t \in [0, T]} \|f(\mathbf{y})\|_{W^{1, \infty}(U)}) \left(\sum_{i=N_s+1}^N \sqrt{\lambda_i} \|b_i(x)\|_{L^\infty(U)} + B_{\mathbb{T}} \right). \end{aligned}$$

for some constant $\mathcal{E} > 0$.

Proof. Consider the solution to Problem 3 $u_{N_s} \in C^0(\Gamma_s; L^2(0, T; V)) \subset L_\rho^2(\Gamma; L^2(0, T; V))$ where the matrix of coefficients $G(\mathbf{y}_s)$ depends only on the variables Y_1, \dots, Y_{N_s} . Thus $\tilde{u}(\mathbf{y}_s)$ satisfies a.e.

$$\int_U \mathbb{E}[(\partial_t u)v] + \mathcal{A}_{N_s}(\mathbf{y}_s; \tilde{u}(\mathbf{y}_s), v) = \mathbb{E}[\tilde{l}(\mathbf{y}_s; v)] \quad \forall v \in L_\rho^2(\Gamma) \otimes L^2(0, T; V)$$

where $\mathcal{A}_{N_s}(\mathbf{y}_s; \tilde{u}, v) := \mathbb{E}[B(\mathbf{y}_s; \tilde{u}, v)]$. From (25) it follows that

$$\begin{aligned} \int_U \mathbb{E}[\partial_t e(\mathbf{y})v + \nabla e(\mathbf{y})^T G(\mathbf{y}_s, \mathbf{y}_f) \nabla v] &= \int_U \mathbb{E}[\nabla \tilde{u}(\mathbf{y}_s)(G(\mathbf{y}_s) - G(\mathbf{y}_s, \mathbf{y}_f)) \nabla v] \\ &+ \mathbb{E}[\tilde{l}(\mathbf{y}_s, \mathbf{y}_f; v) - \tilde{l}(\mathbf{y}_s; v)] \end{aligned}$$

for all $v \in L^2(0, T; L_\rho^2(\Gamma) \otimes V)$, where

$$\begin{aligned} \mathbb{E}[\tilde{l}(\mathbf{y}_s, \mathbf{y}_f; v) - \tilde{l}(\mathbf{y}_s; v)] &= \int_U \mathbb{E}[(f \circ F(\mathbf{y}_s))|\partial F(\mathbf{y}_s)| - f \circ F(\mathbf{y}_s, \mathbf{y}_f)|\partial F(\mathbf{y}_s, \mathbf{y}_f)|]v \\ &+ (\nabla \mathbf{w}^T(\mathbf{y}_s)G(\mathbf{y}_s) - \nabla \mathbf{w}^T(\mathbf{y}_s, \mathbf{y}_f)G(\mathbf{y}_s, \mathbf{y}_f)) \nabla v]. \end{aligned}$$

Now, pick $v = \partial_t e(\mathbf{y})$ and note that $B(\mathbf{y}; e(\mathbf{y}), \partial_t e(\mathbf{y})) = \partial_t(\frac{1}{2}B(\mathbf{y}; e(\mathbf{y}), e(\mathbf{y})))$ then

$$(27) \quad \|\partial_t e(\mathbf{y})\|_{L_\rho^2(\Gamma) \otimes V} + \mathbb{E}\left[\partial_t\left(\frac{1}{2}B(\mathbf{y}; e(\mathbf{y}), e(\mathbf{y}))\right)\right] \leq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.$$

Now, integrating (27) it follows that

$$(28) \quad \begin{aligned} \frac{1}{2}\mathbb{E}[B(\mathbf{y}; e(\mathbf{y}, T), e(\mathbf{y}, T))] + \int_0^T \|\partial_t e(\mathbf{y})\|_{L_\rho^2(\Gamma) \otimes V} &\leq \mathbb{E}[B(\mathbf{y}; e(\mathbf{y}, 0), e(\mathbf{y}, 0))] \\ &+ \int_0^T \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3, \end{aligned}$$

where $\int_0^T \mathcal{A}_1$, $\int_0^T \mathcal{A}_2$ and $\int_0^T \mathcal{A}_3$ are bounded as:

$$\begin{aligned}
(29) \quad \int_0^T \mathcal{A}_1 &:= \|\tilde{u}(\mathbf{y}_s)\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)} \|\partial_t e(\mathbf{y})\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)} \sup_{x \in U, \mathbf{y} \in \Gamma} \|G(\mathbf{y}_s, \mathbf{y}_f) - G(\mathbf{y}_s)\| \\
&\leq \|\tilde{u}(\mathbf{y}_s)\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)} (\|\partial_t \tilde{u}(\mathbf{y})\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)} + \|\partial_t \tilde{u}(\mathbf{y}_s)\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)}) \\
&\quad \sup_{x \in U, \mathbf{y} \in \Gamma} \|G(\mathbf{y}_s, \mathbf{y}_f) - G(\mathbf{y}_s)\|, \\
\int_0^T \mathcal{A}_2 &:= \int_0^T \int_U |\mathbb{E}[(f \circ F(\mathbf{y}_s) - f \circ F(\mathbf{y}_s, \mathbf{y}_f)) | \partial F(\mathbf{y}_s) | \partial_t e(\mathbf{y})] \\
&\quad + \mathbb{E}[f \circ F(\mathbf{y}_s, \mathbf{y}_f) (|\partial F(\mathbf{y}_s)| - |\partial F(\mathbf{y}_s, \mathbf{y}_f)|) \partial_t e(\mathbf{y})]| \\
&\leq (\|\partial_t \tilde{u}(\mathbf{y})\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes L^2(U))} + \|\partial_t \tilde{u}(\mathbf{y}_s)\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes L^2(U))}) \\
&\quad (\mathbb{F}_{max}^d \sup_{\mathbf{y} \in \Gamma, t \in [0,T]} \|f(\mathbf{y})\|_{W^{1,\infty}(U)} \sup_{x \in U, \mathbf{y} \in \Gamma} |F(\mathbf{y}_s) - F(\mathbf{y}_s, \mathbf{y}_f)| \\
&\quad + \|f(\mathbf{y})\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes L^2(U))} \sup_{x \in U, \mathbf{y} \in \Gamma} ||\partial F(\mathbf{y}_s)| - |\partial F(\mathbf{y}_s, \mathbf{y}_f)||),
\end{aligned}$$

and

$$\begin{aligned}
(30) \quad \int_0^T \mathcal{A}_3 &\leq \int_0^T \int_U \mathbb{E}[|\nabla(\mathbf{w}(\mathbf{y}_s) - \mathbf{w}(\mathbf{y}_s, \mathbf{y}_f))^T G(\mathbf{y}_s, \mathbf{y}_f)) \nabla \partial_t e(\mathbf{y})|] \\
&\quad + \mathbb{E}[|\nabla \mathbf{w}^T(\mathbf{y}_s, \mathbf{y}_f) (G(\mathbf{y}_s) - G(\mathbf{y}_s, \mathbf{y}_f)) \nabla \partial_t e(\mathbf{y})|] \\
&\leq (\|\partial_t \tilde{u}(\mathbf{y})\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)} + \|\partial_t \tilde{u}(\mathbf{y}_s)\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)}) (a_{max} \mathbb{F}_{max}^d \mathbb{F}_{min}^{-2} \\
&\quad \sup_{x \in U, \mathbf{y} \in \Gamma} |F(\mathbf{y}_s) - F(\mathbf{y}_s, \mathbf{y}_f)| \|\mathbf{w}(\mathbf{y})\|_{W^{1,\infty}(U)} + \|\mathbf{w}(\mathbf{y})\|_{L^2(0,T;L_\rho^2(\Gamma)\otimes V)} \\
&\quad \sup_{x \in U, \mathbf{y} \in \Gamma} \|G(\mathbf{y}_s, \mathbf{y}_f) - G(\mathbf{y}_s)\|).
\end{aligned}$$

From Theorem 9 in [4] it follows that

$$\begin{aligned}
(31) \quad \sup_{x \in U, \mathbf{y} \in \Gamma} \|G(\mathbf{y}_s, \mathbf{y}_f) - G(\mathbf{y}_s)\| &\leq \|a\|_{W^{1,\infty}(U)} \|\hat{v}\|_{L^\infty(U)} \sum_{i=N_s+1}^N \sqrt{\lambda_i} \|b_i(x)\|_{L^\infty(U)} \\
&\quad + a_{max} B_{\mathbb{T}} H(\mathbb{F}_{max}, \mathbb{F}_{min}, \tilde{\delta}, d)
\end{aligned}$$

where $H(\mathbb{F}_{max}, \mathbb{F}_{min}, \tilde{\delta}, d) := \mathbb{F}_{max}^{d-1} \mathbb{F}_{min}^{-3} (\mathbb{F}_{max}(2 + \mathbb{F}_{min}^{-1}(1 - \tilde{\delta})) + \mathbb{F}_{min}^{-1}d)$. In addition, from Theorem 9 in [4] it follows that

$$(32) \quad \sup_{x \in U, \mathbf{y} \in \Gamma} |F(\mathbf{y}_s) - F(\mathbf{y}_s, \mathbf{y}_f)| \leq |\hat{v}| \sum_{i=N_s+1}^N \sqrt{\lambda_i} \|b_i(x)\|_{L^\infty(U)}$$

and

$$(33) \quad \sup_{x \in U, \mathbf{y} \in \Gamma} ||\partial F(\mathbf{y}_s, \mathbf{y}_f)| - |\partial F(\mathbf{y}_s)|| \leq \mathbb{F}_{max}^{d-1} \mathbb{F}_{min}^{-2} B_{\mathbb{T}} d.$$

Now, from (28) it follows that

$$\begin{aligned}
\|e(\mathbf{y}, T)\|_{L^2_\rho(\Gamma) \otimes V} &\leq \frac{2a_{\max} \mathbb{F}_{\max}^d \mathbb{F}_{\min}^{-2} \|e(\mathbf{y}, 0)\|_{L^2_\rho(\Gamma) \otimes H^1(U)} + 2 \int_0^T \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3}{a_{\min} \mathbb{F}_{\min}^d \mathbb{F}_{\max}^{-2}} \\
&\leq \frac{2a_{\max} \mathbb{F}_{\max}^d \mathbb{F}_{\min}^{-2} \|u_0 \circ F(\mathbf{y}) - u_0 \circ F(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes H^1(U)}}{a_{\min} \mathbb{F}_{\min}^d \mathbb{F}_{\max}^{-2}} \\
(34) \quad &+ \frac{2 \int_0^T \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3}{a_{\min} \mathbb{F}_{\min}^d \mathbb{F}_{\max}^{-2}} \\
&\leq \frac{2a_{\max} \mathbb{F}_{\max}^d \mathbb{F}_{\min}^{-2}}{a_{\min} \mathbb{F}_{\min}^d \mathbb{F}_{\max}^{-2}} \sup_{x \in U, \mathbf{y} \in \Gamma} C(U) \|u_0(\mathbf{y})\|_{W^{1,\infty}(U)} |F(\mathbf{y}) - F(\mathbf{y}_s)| \\
&+ \frac{2 \int_0^T \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3}{a_{\min} \mathbb{F}_{\min}^d \mathbb{F}_{\max}^{-2}},
\end{aligned}$$

where $C(U)$ depends on U . Combining (26), (29) - (34) the result is obtained. \square

5.2. Sparse Grid Error (II). In this section only explicit convergence rates for the isotropic Smolyak sparse grid. Given the bounded linear functional Q it follows that

$$\begin{aligned}
\|Q(e(\mathbf{y}_s))\|_{L^2_\rho(\Gamma_s)} &\leq \left\| \int_U e(\mathbf{y}_s) \partial_t \varphi(\mathbf{y}_s) + B(\mathbf{y}_s; e(\mathbf{y}_s), \varphi(\mathbf{y}_s)) \right\|_{L^2_\rho(\Gamma)} \\
&\leq \|\partial_t \varphi(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes L^2(U)} \|e(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes L^2(U)} \\
&+ a_{\max} \mathbb{F}_{\max}^d \mathbb{F}_{\min}^{-2} \|\varphi(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes V} \|e(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes V} \\
&\leq \|e(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes V} (C_P(U) \|\partial_t \varphi(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes L^2(U)} \\
&+ a_{\max} \mathbb{F}_{\max}^d \mathbb{F}_{\min}^{-2} \|\varphi(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes V}) \\
&\leq C(U, T, G(\mathbf{y})) T \|q\|_{L^2_\rho(\Gamma) \otimes L^2(U)} \|e(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes V} (C_P(U) \\
&+ a_{\max} \mathbb{F}_{\max}^d \mathbb{F}_{\min}^{-2}),
\end{aligned}$$

where $e(\mathbf{y}_s) := \tilde{u}(\mathbf{y}_s) - \mathcal{S}_w^{m,g}[\tilde{u}(\mathbf{y}_s)]$. The last inequality and dependence of $C(U, T, G(\mathbf{y})) > 0$ are obtained from Theorem 5 in [6]. The next step is to bound the term $\|e(\mathbf{y}_s)\|_{L^2_\rho(\Gamma) \otimes V}$.

As noted in Section 4.1, the sparse grid is computed with respect to the auxiliary density function $\hat{\rho}$, thus

$$\|e\|_{L^2_\rho(\Gamma) \otimes V} \leq \left\| \frac{\rho}{\hat{\rho}} \right\|_{L^\infty(\Gamma_s)} \|e\|_{L^2_{\hat{\rho}}(\Gamma) \otimes V}.$$

The goal now is to control the error term $\|e\|_{L^2_{\hat{\rho}}(\Gamma_s) \otimes V}$. This term is directly dependent on i) the number of collocation knots η (or work), ii) the choice of the approximation formulas $(m(\mathbf{i}), g(\mathbf{i}))$, iii) the choice of abscissas and iv) the region of analyticity of $\Theta_\beta \subset \mathbb{C}^{N_s}$. From Theorem 1 the solution $\tilde{u}(\mathbf{y}_s)$ admits an extension in \mathbb{C}^{N_s} i.e. $\tilde{u}(\mathbf{z}_s) \in C^0(\Theta_\beta; L^2(0, T; V))$.

In [16, 17] the authors derive error estimates for isotropic and anisotropic Smolyak sparse grids with Clenshaw-Curtis and Gaussian abscissas where $\|e\|_{L^2_{\hat{\rho}}(\Gamma_s; V)}$ exhibit algebraic or sub-exponential convergence with respect to the number of collocation knots η (See Theorems 3.10, 3.11, 3.18 and 3.19 for more details). However, for these estimates to be valid the solution \tilde{u} has

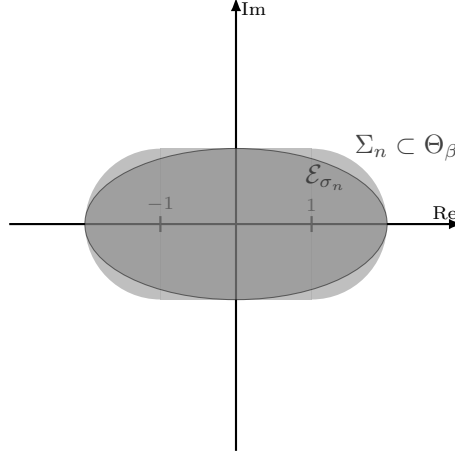


FIGURE 3. Embedding of the polyellipse $\mathcal{E}_{\sigma_1, \dots, \sigma_{N_s}} := \Pi_{n=1}^{N_s} \mathcal{E}_{n, \sigma_n}$ in $\Sigma \subset \Theta_\beta$. Each ellipse $\mathcal{E}_{n, \sigma_n}$ is embedded in $\Sigma_n \subset \Theta_\beta$ for $n = 1, \dots, n$.

to admit and extension on a polyellipse in \mathbb{C}^{N_s} , $\mathcal{E}_{\sigma_1, \dots, \sigma_{N_s}} := \Pi_{i=1}^{N_s} \mathcal{E}_{n, \sigma_n}$, where

$$\mathcal{E}_{n, \sigma_n} := \left\{ z \in \mathbb{C}; \operatorname{Re}(z) = \frac{e^{\sigma_n} + e^{-\sigma_n}}{2} \cos(\theta), \operatorname{Im}(z) = \frac{e^{\sigma_n} - e^{-\sigma_n}}{2} \sin(\theta), \theta \in [0, 2\pi) \right\},$$

and $\sigma_n > 0$. For an isotropic sparse grid the overall asymptotic subexponential decay rate $\hat{\sigma}$ will be dominated by the smallest σ_n i.e. $\hat{\sigma} \equiv \min_{n=1, \dots, N_s} \sigma_n$. The polyellipse is then embedded in Θ_β . Consider the region $\Sigma \subset \mathbb{C}^{N_s}$ such that $\Sigma \subset \Theta_\beta$, where $\Sigma := \Sigma_1 \times \dots \times \Sigma_{N_s}$ and

$$\Sigma_n := \left\{ \mathbf{z} \in \mathbb{C}; \mathbf{z} = \mathbf{y} + \mathbf{v}, \mathbf{y} \in [-1, 1], |v_n| \leq \tau_n := \frac{\beta}{1 - \delta} \right\},$$

for $n = 1, \dots, N_s$. For the choice $\sigma_1 = \sigma_2 = \dots = \sigma_{N_s} = \hat{\sigma} = \log(\sqrt{\tau_{N_s}^2 + 1} + \tau_{N_s}) > 0$ it can be seen that the polyellipse $\mathcal{E}_{\sigma_1, \dots, \sigma_{N_s}}$ can now be embedded in Σ , as shown in Figure 3.

Given a sufficiently large of collocation points η with a CC sparse grid Theorem 3.11 in [17] states that

$$(35) \quad \|e\|_{L^2_\beta(\Gamma_s; V)} \leq \mathcal{Q}(\sigma, \delta^*, N_s) \eta^{\mu_3(\sigma, \delta^*, N_s)} \exp\left(-\frac{N_s \sigma}{2^{1/N_s}} \eta^{\mu_2(N_s)}\right)$$

where $\mathcal{Q}(\sigma, \delta^*, N_s) := \frac{C_1(\sigma, \delta^*)}{\exp(\sigma \delta^* C_2(\sigma))} \frac{\max\{1, C_1(\sigma, \delta^*)\}^{N_s}}{|1 - C_1(\sigma, \delta^*)|}$, $\sigma = \hat{\sigma}/2$, $\mu_2(N_s) = \frac{\log(2)}{N_s(1 + \log(2N_s))}$ and $\mu_3(\sigma, \delta^*, N_s) = \frac{\sigma \delta^* \tilde{C}_2(\sigma)}{1 + \log(2N_s)}$. The constants $C_1(\sigma, \delta^*)$, $\tilde{C}_2(\sigma)$ and δ^* are defined in [17] eqns (3.11) and (3.12).

6. COMPLEXITY AND TOLERANCE

In this section the total work W is analyzed such that $|\operatorname{var}[Q(\mathbf{y}_s, \mathbf{y}_f)] - \operatorname{var}[\mathcal{S}_w^{m, g}[Q_h(\mathbf{y}_s)]]|$ and $|\mathbb{E}[Q(\mathbf{y}_s, \mathbf{y}_f)] - \mathbb{E}[\mathcal{S}_w^{m, g}[Q_h(\mathbf{y}_s)]]|$ for the isotropic CC sparse grid is less or equal to a given tolerance parameter $\operatorname{tol} \in \mathbb{R}^+$. Now, let $\mathcal{S}_w^{m, g}$ be the sparse grid operator characterized by $m(i)$ and $g(\mathbf{i})$. The total work for computing the variance $\mathbb{E}[(\mathcal{S}_w^{m, g}[Q_h(\mathbf{y}_s)])^2] - \mathbb{E}[\mathcal{S}_w^{m, g}[Q_h(\mathbf{y}_s)]]^2$ and

the mean term $\mathbb{E}[\mathcal{S}_w^{m,g}[Q_h(\mathbf{y}_s)]]$ with respect to the given user tolerance is

$$W_{Total}(tol) = W_{sol}(tol)\eta(tol),$$

where $\eta(N_s, m, g, w, \Theta_\beta)$ is the number of the sparse grid points and W_{sol} is the work needed to compute each deterministic solve.

- (a) **Truncation:** From the truncation estimate derived in section 5.1 it is sought that $\|Q(\mathbf{y}_s, \mathbf{y}_f) - Q(\mathbf{y}_s)\|_{L_\rho^2(\Gamma)} \leq \frac{tol}{3C_T}$ with respect to the decay of λ_i . First, make the assumption that $B_\mathbb{T} = \sup_{x \in U} \sum_{i=N_s+1}^N \sqrt{\lambda_i} \|B_i(x)\|_2 \leq C_D N_s^{-l}$ for some uniformly bounded $C_D > 0$. Furthermore, assume that $\|b_i(x)\|_{L^\infty(U)} \leq D_D \sup_{x \in U} \|B_i(x)\|_2$ for $i = 1, \dots, N$ where $D_D > 0$ is uniformly bounded, thus $\tilde{B}_\mathbb{T} := \sup_{x \in U} \sum_{i=N_s+1}^N \sqrt{\lambda_i} \|b_i(x)\|_2 \leq C_D D_D N_s^{-l}$. It follows that $\|Q(\mathbf{y}_s, \mathbf{y}_f) - Q(\mathbf{y}_s)\|_{L_\rho^2(\Gamma)} \leq \frac{tol}{3C_T}$ if $\tilde{B}_\mathbb{T} + B_\mathbb{T} \leq C_D(1 + D_D)N_s^{-l} \leq \mathcal{E}tol$ for some constant $\mathcal{E} > 0$ with the same dependence as in Theorem 2. Finally,

$$N_s(tol) \geq \left(\frac{\mathcal{E}tol}{C_D(1 + D_D)} \right)^{-1/l}.$$

- (b) **Sparse Grid:** Following the same strategy as in [17] (eqn (3.39)), to simplify the bound (35) choose $\delta^* = (e \log(2) - 1)/\tilde{C}_2(\sigma)$ and $\tilde{C}_2(\sigma)$. Thus $\|e\|_{L_\rho^2(\Gamma_s; V)} \leq \frac{tol}{3C_{SG}C_T} \left\| \frac{\rho}{\hat{\rho}} \right\|_{L^\infty(\Gamma_s)}^{-1}$ if

$$\eta(tol) \geq \left(\frac{3\|\rho/\hat{\rho}\|_{L^\infty(\Gamma_s)} C_{SG} C_T C_F F^{N_s} \exp(\sigma(\beta, \tilde{\delta}))}{tol} \right)^{\frac{1+\log(2N_s)}{\sigma}}$$

where $C_F = \frac{C_1(\sigma, \delta^*)}{|1 - C_1(\sigma, \delta^*)|}$ and $F = \max\{1, C_1(\sigma, \delta^*)\}$.

Combining (a) and (b) it follows that for a given user error tolerance tol the total work is

$$\begin{aligned} W_{Total}(tol) &= W_{sol} \eta(\tilde{\delta}, \beta, N_s(tol), \|\rho/\hat{\rho}\|_{L^\infty(\Gamma_s)}, C_{SG}, C_T, C_F, F) \\ &= \mathcal{O} \left(W_{sol} \left(\frac{\|\rho/\hat{\rho}\|_{L^\infty(\Gamma_s)} F^{C_{tol}^{-1/l}}}{tol} \right)^{\sigma^{-1}(1+l^{-1}(\log C_D(1+D_D) - \log 2\mathcal{E}tol))} \right). \end{aligned}$$

7. NUMERICAL RESULTS

In this section a numerical example is shown that compares the the mean and variance of the QoI with the derived Smolyak sparse grid convergence rates.

Suppose the reference domain is set $U = (0, 1) \times (0, 1)$ and deforms according to the following rule:

$$\begin{aligned} F(x_1, x_2) &= (x_1, (x_2 - 0.5)(1 + ce(\omega, x_1)) + 0.5) & \text{if } x_2 > 0.5 \\ F(x_1, x_2) &= (x_1, x_2) & \text{if } 0 \leq x_2 \leq 0.5 \end{aligned}$$

for some positive constant $c > 0$ i.e. only the upper half of the domain and fix the bottom half. The Dirichlet boundary conditions are set to zero for the upper border. The rest of the borders are set to Neumann boundary conditions with $\frac{\partial u}{\partial \nu} = 1$ (See Figure 4 (a)). In Figure 4 (b) a realization of the reference domain under this deformation model and boundary conditions.

To solve the parabolic PDE a finite element discretization is used for the spatial and a backward Euler method with a step size of $t_d = 1/400$ and final time $T = 1$. In Figure 4 (c) the contours of the solution for the realization (b) for the final time are shown.

The QoI is defined on the bottom half of the reference domain, which is not deformed, as $Q(\tilde{u}(T)) := \int_{(0,1)} \int_{(0,1/2)} g(x_1)g(2x_2)\tilde{u}(\omega, x_1, x_2, T) dx_1 dx_2$. To make a direct comparison between the Smolyak theoretical decay rates and the numerical results the gradient terms $\sqrt{\lambda_n} \sup_{x \in U} \|B_n(x)\|$ are set to decay linearly as n^{-1} . Furthermore the following parameters and functions are set as:

- $a(x) = 1$ for all $x \in U$.
- Stochastic Model: $e_S(\omega, x_1) := Y_1(\omega) \left(\frac{\sqrt{\pi}L}{2} \right) + \sum_{n=2}^{N_s} \sqrt{\lambda_n} \varphi_i(x_1) Y_n(\omega)$; $e_F(\omega, x_1) := \sum_{n=N_s+1}^N \sqrt{\lambda_n} \varphi_n(x_1) Y_n(\omega)$.
- Linear decay $\sqrt{\lambda_n} := \frac{(\sqrt{\pi}L)^{1/2}}{n}$, $n \in \mathbb{N}$, $\varphi_i(x_1) := \begin{cases} n^{-1} \sin\left(\frac{|n/2|\pi x_1}{L_p}\right) & \text{if } n \text{ is even} \\ n^{-1} \cos\left(\frac{|n/2|\pi x_1}{L_p}\right) & \text{if } n \text{ is odd} \end{cases}$

This implies that $\sup_{x \in U} \sigma_{\max}(B_l(x))$ is bounded by a constant and linear decay on the gradient of the deformation is obtained.

- $\{Y_n\}_{n=1}^N$ are independent uniform distributed in $(-\sqrt{3}, \sqrt{3})$.
- $L = 19/50$, $L_p = 1$, $c = 1/2.175$, $N = 15$.
- The stochastic domain is represented with a triangular mesh 129×129 .
- $\mathbb{E}[Q_h]$ and $\text{var}(Q_h)$, are computed with a Clenshaw-Curtis isotropic sparse grid (Sparse Grid Toolbox V5.1, [12, 11]).
- The reference solutions $\text{var}[Q_h(u_{ref})]$ and $\mathbb{E}[Q_h(u_{ref})]$ are computed with a dimension adaptive Sparse Grid ($\approx 30,000$ knots) [8] with a 127×127 mesh for $N = 15$ dimensions.
- The QoI is normalized by the solution of the reference domain $Q_h(U)$.

In Figure 5 the results of the matlab code are shown for $N_s = 2, 3, 4$ and compare the results with respect to a $N = 15$ dimensional adaptive sparse grid method collocation with $\approx 30,000$ collocation points [8]. The computed mean value is 0.9846 and variance is 0.0342 (0.1849 std).

In Figure 5 (a) and (b) the normalized mean and variance errors are shown for $N_s = 2, 3, 4$. For (a) notice that the mean quickly saturates by the truncation error and/or finite element error starts to dominate. In (b) the variance error decay is subexponential as indicated by the error bounded in (35) until truncation/finite element errors dominate.

The truncation error will be now analyzed. For $N_s = 2, 3, 4$ the mean and variance are computed as in (g). However, for $N_s = 5, 6, 7, 8, 9$ a dimensional adaptive sparse grid is used with 10,000 to 30,000 sparse grid points. This is to minimize the influence of the sparse grid on the truncation error. Note, that the reference solution for the mean and variance is computed as in part (h).

In Figure 6 the truncation error is plotted for (a) the mean and (b) the variance with respect to the number of dimensions. From these plots observe that the convergence rates is close to quadratic, which is at least one order of magnitude higher than the derived truncation convergence rate.

8. CONCLUSIONS

In this paper a rigorous convergence analysis is derived for a sparse grid stochastic collocation method for the numerical solution of parabolic PDEs with random domains. The following contributions are achieved in this work:

- An analysis of the regularity of the solution with respect to the parameters describing the domain perturbation show that an analytic extension into a well defined region Θ_β embedded in \mathbb{C}^N exists.
- Error estimates in the energy norm and the QoI for the Clenshaw Curtis abscissas are derived. From the analytic regularity extension of the solution in Θ_β a subexponential convergence is achieved that is consist with the numerical results.
- A truncation error with respect to the

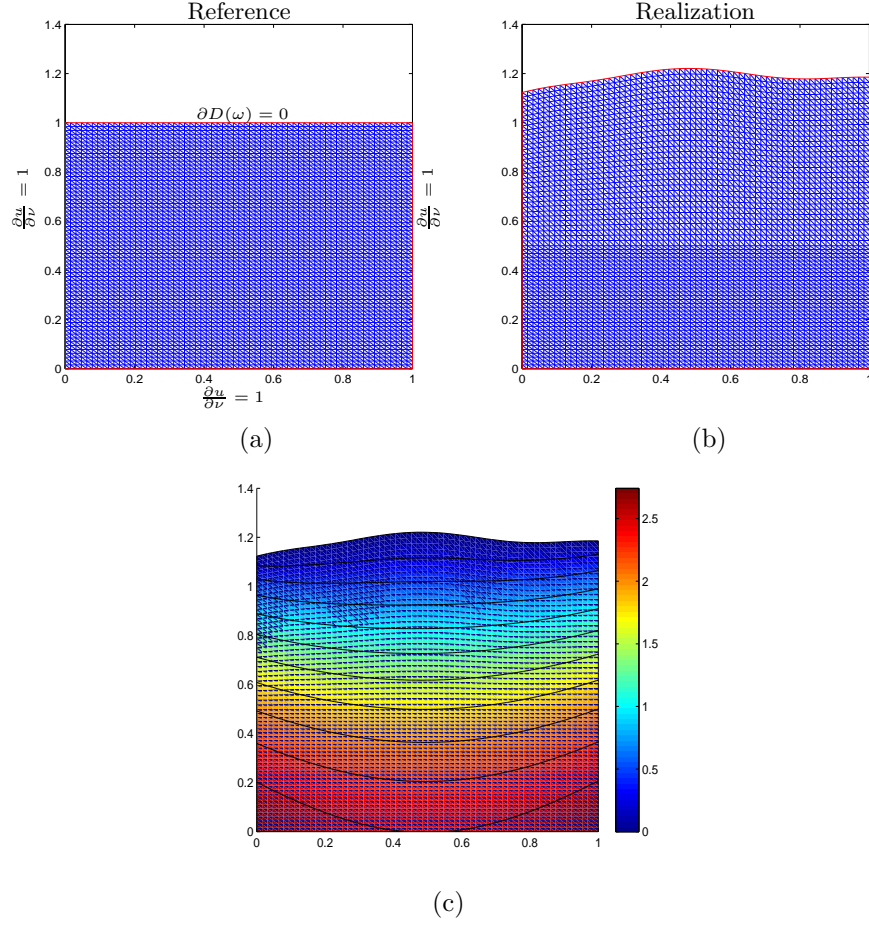


FIGURE 4. Stochastic deformation of a square domain and solution on a realization of the stochastic domain. (a) Reference square domain with Dirichlet boundary conditions. (b) Vertical deformation from stochastic model. (c) Contours of the solution of the parabolic PDE for $T = 1$ on the stochastic deformed domain realization.

number of random variables is derived. Numerical experiments are consistent with the derived convergence rate. **iv)** A complexity formula is derived for the total amount of work needed to achieve a given accuracy tolerance.

The approach described in this paper is limited to Isotropic sparse grids, thus limiting applicability to moderate number of stochastic variables. However, it is not hard to see that it will also be applicable to anisotropic sparse grids, thus significantly increasing the practicality of this method with respect to a larger N . This, however, will be left as future work.

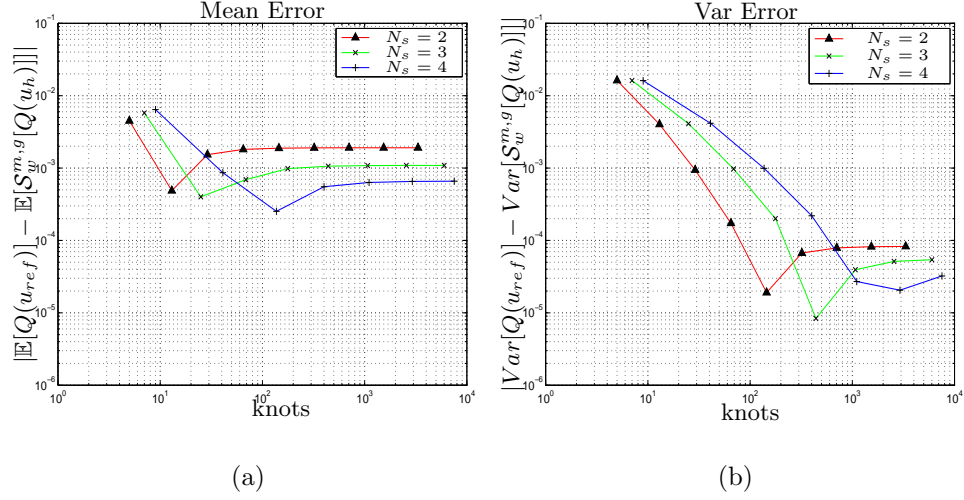


FIGURE 5. Collocation results for $N_s = 2, 3, 4$ with linear decay. (a) Mean error with respect to reference. Observe that the convergence rate decays subexponentially until truncation saturation is reached. (b) Variance error for with respect to reference. For this case observe that the convergence rate is also subexponential.

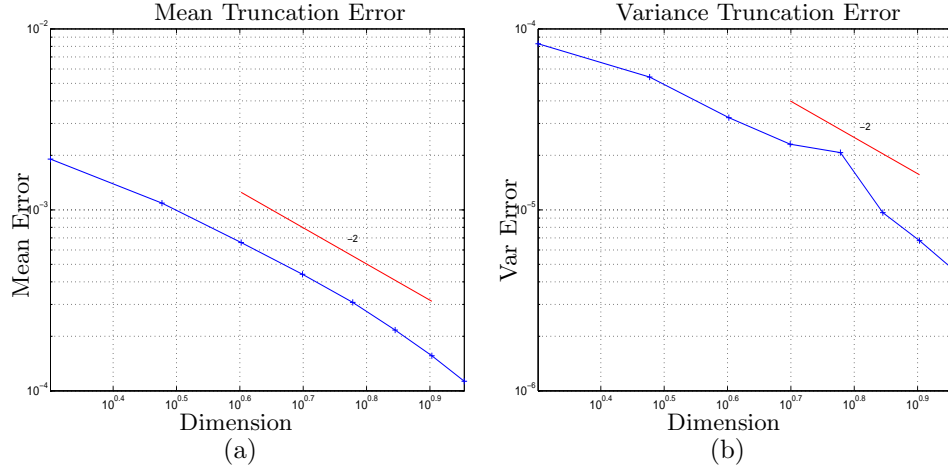


FIGURE 6. Truncation Error with respect to the number of dimension. (a) Mean error. (b) Variance error. In both cases the decay appears faster than linear, which is faster than the predicted convergence rate.

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